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Statistical-Physical Models of Man-Made and Natural Radio Noise

PART III. FIRST-ORDER PROBABILITY MODELS OF THE INSTANTANEOUS AMPLITUDE OF CLASS B INTERFERENCE

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PART III. FIRST-ORDER PROBABILITY MODELS OF THE INSTANTANEOUS AMPLITUDE OF CLASS B INTERFERENCE

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U.S. DEPARTMENT OF COMMERCE
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PREFACE

This Report is the third (i.e. Part III) in a series of ongoing studies [Middleton, 1974, 1976] of the general electromagnetic (EM) interference environment arising from man-made and natural EM noise sources, and is also part of the continuing analytical and experimental effort whose general aims are [Spaulding and Middleton, 1975, 1978]:

(1). to provide quantitative, statistical descriptions of man-made and natural electromagnetic interference (as in this series);

(2). to indicate and to guide experiment, not only to obtain pertinent data for urban and other EM environments, but also to generate standard procedures and techniques for assessing such environments;

(3). to determine and predict system performance in these general electromagntic milieux, and to obtain and evaluate optimal system structures therein, for

   (a). the general purposes of spectrum management;

   (b). the establishment of appropriate data bases thereto; and

   (c). the analysis and evaluation of large-scale telecommnication systems.

With the aid of (1) and (2) one can predict the interference characteristics in selected regions of the electromagnetic spectrum, and with the results of (3), rational criteria of performance can be developed to predict the successful or unsuccessful operation of telecommunication links and systems in various classes of interference. Thus, the combination of the results of (1)-(3) provide specific, quantitative procedures for spectral management, and a reliable technical base for the choice and implementation of policy decisions thereto.

The man-made EM environment, and most natural EM noise sources as well, are basically "impulsive", in the sense that the emitted waveforms have a highly structured character, with significant probabilities of large interference levels. This is noticeably different from the usual normal (gaussian) noise processes inherent in transmitting and receiving elements. This highly structured character of the interference can drastically degrade performance of conventional systems, which are optimized, i.e. designed to operate most effectively, against the customarily assumed normal background noise processes. The present Report is devoted to the problems of (1), (2) above, namely, to provide adequate statistical physical models, verified by experiment, of these general "impulsive", highly non-gaussian interference processes, which constitute a principal corpus of the interference environment, and which are required in the successful pursuit of (3), as well. The principal new results here are:

(i). Canonical analytical models for the first-order statistics of the instantaneous amplitude and its magnitude for Class B noise*;

(ii). Procedures for obtaining the (canonical) model parameters, from the APD's (= exceedance probabilities, $P_1(\mathcal{E}>\mathcal{E}_0)$, etc.), the calculation of moments, and probability density functions (pdf's), and a variety of other related statistics; [see the Table of Contents].

Finally, we emphasize, again, that it is the quantitative interplay between the experimentally established,** analytical model-building for the electromagnetic environment, and the evaluation of system performance therein, which provides essential tools for prediction and performance, for the development of adequate, appropriate data bases, procedures for effective standardizations, and spectrum assessment, required for the effective management of the spectral-use environment.

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* Class A and Class B noise are distinguished, qualitatively, by having input bandwidths which are respectively narrower and broader than that of the (linear) front-end stages of the typical (narrow-band) receiver in use. More precise definitions are developed in the text following.

** Experimental corroboration has been achieved on the basis of envelope data for both the Class A and B interference processes [cf. Middleton, 1976, Section 2.4]. An equivalent corroboration for the corresponding amplitude data is accordingly inferred.

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STATISTICAL-PHYSICAL MODELS OF MAN-MADE AND NATURAL RADIO NOISE

Part III. First-Order Probability Models of the Instantaneous Amplitude of Class B Interference*

by

David Middleton**

ABSTRACT

This Report is the third ("Part III") in a continuing series devoted to the development of analytically tractable, statistical-physical models of man-made and natural electromagnetic interference. Here, the first-order statistical probability densities (pdf's) and the associated exceedance probabilities (PD's, or APD's) are obtained for the instantaneous amplitudes (X), and instantaneous magnitudes, |X|, of Class B noise. These are needed not only for experimental studies but, also, particularly for the analysis and evaluation of the performance of optimum and suboptimum receivers in Class B interference environments.

As in the earlier studies of the envelope statistics of Class B noise [Middleton, 1976], a two-function approximation is needed for the characteristic function and hence for the corresponding pdf's and PD's. Two methods of determining the six (basic) parameters which describe these first-order statistics and thus joining the approximate forms (pdf's and PD's) are outlined. Method 1 is approximate, was used earlier [Middleton, 1976, 1977], and has the advantage of somewhat greater computational simplicity, with the disadvantage, however, of yielding too low values of the PD at low values of the argument (X), when the gaussian component is small. Method 2 is "exact", and somewhat more complex computationally. The joining process involved in both methods has been essentially described earlier [Middleton,

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* This is the third in a continuing series of Reports developing the first-order (and later, various higher-order) statistics of natural and man-made radio noise. Earlier studies in this series are: Part I-OT Report 74-36, April, 1974; Part II-OT Report 76-86, April, 1976.

** The author is under contract with the U.S. Department of Commerce, National Telecommunications and Information Administration, Institute for Telecommunication Sciences, Boulder, Colorado 80303.

1976, 1977] but is developed further here. The basic parameters are, in any case, the same as those derived for the envelope statistics. The excellent agreement with experiment observed for the envelope data accordingly applies here, as well.

STATISTICAL-PHYSICAL MODELS OF MAN-MADE AND NATURAL RADIO NOISE

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PART I: INTRODUCTION, RESULTS AND CONCLUSIONS

1. Introduction:

Again, as in earlier studies (for example, [Middleton, 1972a, 1972b, 1973, 1974, 1976, 1977]), our central problem here is to construct analytically tractable models of man-made and natural radio noise, based on the pertinent statistical-physics and possessing a canonical structure, invariant of any particular source mechanisms. This is done for three principal reasons:

(i). To provide realistic, quantitative descriptions of man-made and natural electromagnetic (EM) interference environments;

(ii). To specify and guide experiments for measuring such interference environments; and,

(iii). To determine the structure of optimal communication systems and to evaluate and compare their performance with that of specified suboptimum systems, when operating in these general classes of EM interference.

These three tasks, in turn, are central features of any adequate program of spectrum management (for example, [Middleton, 1975a]).

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Accordingly, our aim, as in previous studies in this series [cf. Middleton, Part I, 1974; Part II, 1976] is to provide analytical models, which:

(1). combine the appropriate physical and statistical descriptions of general EM interference environments;
(2). are analytically manageable;
(3). possess canonical properties, i.e., are not specialized to individual noise mechanisms, source distributions, and emission waveforms, propagation laws, etc.; and, most important,
(4). are experimentally verifyable and predictive.

In addition, the basic parameters of such statistical-physical models must be measurable quantities with a specified physical structure and interpretation.

To achieve the above is clearly a non-trivial-problem, as we have noted before [Middleton, 1972a, 1974, 1976], chiefly because of the inherent highly nongaussian nature of these random processes, a characteristic which at once insures complex descriptions and ensuing difficulties for the analysis of system performance. That these difficulties can be effectively overcome for model-building and experimental verification has been demonstrated by recent efforts [Middleton, 1976, 1977]. For receiver design and performance, using these models, we cite the recent work of Spaulding and Middleton [1975, 1977], which establishes the applicability of the canonical models to receiver evaluation in these EM environments.

Specifically, earlier work here has focused on amplitude and envelope statistics: we note

(i). Middleton, [1974]: First-order statistics of the instantaneous amplitude (X), Class A noise;
(ii). Middleton, [1976]: First-order statistics of the envelope (E) and phase (ϕ), Class A and Class B noise;

The present effort (Part III, 1977) is devoted to the corresponding statistics of the instantaneous amplitude (X), for Class B interference, while Part IV (in preparation) extends the above to the general Class C (= Class A + Class B) environments.* Still earlier, important related studies have been carried out (essentially for what is now called Class B noise) by Furutsu and Ishida [1960], and Giordano [1970, 1972] and Haber [1972], which, however, are not canonical and which focus primarily on atmospheric noise. The principal new results obtained here are the probability distributions (PD's) and probability densities (pdf's), moments, and characteristic functions (c.f.'s) for the Class B instantaneous amplitudes (cf. analytical details in Part II ff.). The analysis and approach here is structured directly upon that of Part II [Middleton, 1976], with appropriate modifications. Thus, the present Report is organized as follows: Section 2 contains a concise account of the principal results; Sections 3-7 comprise the body of Part II here, viz., the analytical details. These are: Section 3, an introduction to the analysis; Section 4, integral forms for the PD, $P_1(X \geq X_0)$ and pdf, $w_1(X)$, for Class A, B, and C interference; Section 5, the PD's, $P_1(X \geq X_0)$ for Class B noise, with particular attention to an (earlier) approximate method (Method 1) of obtaining the distribution parameters, and an exact method (Method 2), which is pertinent for all Class B models, while Section 6 gives the pdf's, $w_1(X)_B$. Section 7 completes the Report with remarks on moments and model parameters.

2. Results and Conclusions:

Let us now summarize the principal results of this study in a concise fashion. The analytic details and representative calculations (and Figures) for the PD's and pdf's are developed and presented in Part II following. The main analytic results are:

Characteristic functions (c.f.'s):

$$\left.\hat{F}_1(ia\lambda)_{B-I} \equiv e^{-b_{1\alpha}A_B\hat{a}^{\alpha}|\lambda|^{\alpha}-\Delta\sigma_G^2\hat{a}^2\lambda^2/2} \quad \begin{array}{l}(\doteq F_1(ia\lambda)_{B-I}),\ |z_0|,\ |z|\leq|z_{0B}|, \\ \text{(Eq. (5.4a)}\end{array}\right\}, \tag{2.1a}$$

----

* See Section 1.1 (Part I), Middleton [1976], or Middleton [1977] for detailed definitions of Class A, B, and C noise (as seen by the receiver). Essentially, Class A sources are characterized by having emission spectra narrower than the receiver bandpass, while Class B sources are spectrally broader than the receiver bandpass.

$$\hat{F}_1(ia\lambda)_{B-II} \equiv e^{-A_B} \exp[-\sigma_G^2\hat{a}^2\lambda^2/2+A_B e^{-b_{2\alpha}\hat{a}^2\lambda^2/2}] \; (\doteq F_1(ia\lambda)_{B-II}),$$

$$|z_o|,\ |z|\geq|z_{oB}|,\ \text{Eq. (5.4b)}, \tag{2.16}$$

where $\hat{a} \equiv \{\Omega_{2B}(1+\Gamma_B')\}^{-1/2}$, cf. (5.2). Here z, z_o, z_{oB} are normalized amplitudes, e.g.

$$z = X/\sqrt{\Omega_{2B}(1+\Gamma_B')} \;;\; z_{o,B} = X_{o,B}/\sqrt{\Omega_{2B}(1+\Gamma_B')} \;,\ \text{Eq. (5.1)}. \tag{2.1c}$$

The corresponding PD's are

$$P_1(z\geq z_o) = P_1(z\geq z_o)_{\substack{B-I:\ \{-z_{oB}<z_o<z_{oB}\} \\ B-II:\ \{z_o<-z_{oB},\, z_o>z_{oB}\}}} \tag{2.2a}$$

$$P_1(|z|\geq|z_o|) = P_1(|z|\geq|z_o|)_{\substack{B-I:\ 0<|z_o|\leq z_{oB}(>0) \\ B-II:\ |z_o|>z_{oB}}} . \tag{2.2b}$$

With ${}_1F_1$ and Θ respectively the usual confluent hypergeometric function and error integral, we have from (5.11) for the P_{1-B-I}:

$$P_1(z\geq z_o)_{B-I} \equiv \hat{P}_1(\hat{z}\geq\hat{z}_o)_I \simeq \frac{1}{2} - \frac{\hat{z}_o}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Gamma\left(\frac{n\alpha+1}{2}\right)\hat{A}_\alpha^n \cdot {}_1F_1\left(\frac{n\alpha+1}{2};\ 3/2;-\hat{z}_o^2\right) , \tag{2.3a}$$

$$P_1(|z|\geq|z_o|)_{B-I} \equiv \hat{P}_1(|\hat{z}|>|\hat{z}_o|)_I$$

$$\simeq 1-\frac{2|\hat{z}_o|}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Gamma\left(\frac{n\alpha+1}{2}\right)\hat{A}_\alpha^n \;\; {}_1F_1\left(\frac{n\alpha+1}{2};\frac{3}{2};-|\hat{z}_o|^2\right), \tag{2.3b}$$

with $0 \leq |z_o|$, $|z| < |z_{oB}|$ for both.

For the P_{1-B-II}, we have from (5.19):

$$P_1(z>z_o)_{B-II} \simeq \frac{1}{2}\{1-e^{-A_B}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\,\Theta[z_o/\sqrt{2}\,\hat{c}_{mB}]\}, \quad (2.4a)$$

$$P_1(|z|\geq|z_o|)_{B-II} \sim \{1-e^{-A_B}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\,\Theta[|z_o|/\sqrt{2}\,\hat{c}_{mB}]\}, \quad (2.4b)$$

$$|z_o| \geq |z_{oB}|(>0).$$

Here, specifically, we have

$$\hat{A}_\alpha = A_\alpha/2^\alpha G_B^\alpha \;;\; \hat{z}_{oB} = z_{o,B}N_I/2\sqrt{2}\,G_B \;;\; G_B^2 = \frac{\Gamma_B'+\frac{4-\alpha}{2-\alpha}}{4(1+\Gamma_B')}, \text{ Eq. (5.20b)};$$

$(|z_{oB}|>0$, cf. p. 37 .)

$$(2\hat{\sigma}_{mB}^2=)\;\hat{c}_{mB}^2 = (m/\hat{A}_B+\Gamma_B')/(1+\Gamma_B') \;;\; \hat{A}_B = \left(\frac{2-\alpha}{4-\alpha}\right)A_B, \text{ cf. (5.7c)};$$

[A_α, cf. Eq. (5.11c)
α , cf. Eq. (4.24b)].

(2.5)

The associated pdf's, $w_1(z)_B$, are from (6.7) and (6.8):

$$w_1(z)_{B-I} \equiv \hat{w}_1(\hat{z})_I \simeq \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\hat{A}_\alpha^n\Gamma\left(\frac{n\alpha+1}{2}\right)\,{}_1F_1\left(\frac{n\alpha+1}{2};\frac{1}{2};-\hat{z}^2\right)$$

$$(-\hat{z}_{oB}<\hat{z}<\hat{z}_{oB}),\; \hat{z} = zN_I/2\sqrt{2}\,G_B$$

$$\hat{z}_{oB} = z_{oB}N_I/2\sqrt{2}\,G_B$$

(2.6a)

$$w_1(z)_{B-II} \cong e^{-A_B} \sum_{m=0}^{\infty} \frac{A_B^m}{m!} \frac{e^{-z^2/2\hat{c}_{mB}^2}}{\sqrt{2\pi\hat{c}_{mB}^2}} \quad ; (z < -z_{oB}, \; z > z_{oB}) \tag{2.5b}$$

$$w_1(|z|)_{B-I,II} = 2w_1(z)_{B-I,II}, \quad z \geq 0. \tag{2.5c}$$

Figures 5.1, 5.2; 5.3, 5.4 are respectively analogous to Figs. 3.3, 3.4, and 3.1, 3.2 [Middleton, 1976] and are calculated : Figs. 5.1, 5.2 from Eq. (2.3) above [or Eq. (5.11)]; Figs. 5.3, 5.4 from Eq. (2.4) above [or Eqs. (5.17), (5.19)]. The composite PD and pdf, e.g.

$$\left.\begin{aligned} P_1(|z|\geq|z_o|) &= P_1(z\geq z_o)_{B-I} = \hat{P}_1(|\hat{z}|\geq|\hat{z}_o|)_I \;, \quad (0\leq|z|,|z_o|<|z_{oB}|) \;; \\ &= P_1(|z|\geq|z_o|)_{B-II} \;, \quad |z_o|\geq|z_{oB}|(>0) \end{aligned}\right\} \tag{2.7a}$$

and

$$\left.\begin{aligned} w_1(z)_B &= \hat{w}_1(\hat{z})_I \;, \; 0 \leq |\hat{z}| < |\hat{z}_{oB}| \\ &= w_1(z)_{B-II} \;, \; |z| > |z_{oB}| \end{aligned}\quad ,\right\} \tag{2.7b}$$

are illustrated, respectively, in Figs. 5.6, 5.7 [analogous to Figs. 3.6, 3.7, Middleton [1976]], and in Figs. 6.3, 6.2, which are likewise analogous to Figs. 4.3, 4.4 [Middleton,1976]. The composite PD's exhibit the characteristic "bend-over" at the smaller probabilities (and larger arguments, $|z_o|$, etc.). Fig. (5.5) illustrates the composite, approximating procedure schematically, cf. Sec. (5.2)ff. For small $|z_o|$ the gaussian character of the PD's is exhibited, while for the larger values (and correspondingly small values of P_1) the strong departures from gauss appear. Similar observations are noted for the pdf's, Figs. 6.1, 6.2.

The global and generic parameters for these amplitude distributions are precisely the same as for the envelope distributions studied earlier [Middleton, 1976, Secs. 2.2, 2.3, 2.5], in these Class B cases. We review them concisely, for convenience, here:

(1). A_B = the Impulsive Index, for the "Class A-form" of approximation ($|z|>z_{oB}$), used in the overall Class B model. The Impulsive Index is a measure of the nongaussian nature of the noise: the smaller A_B, the more "impulsive" the interference;

(2). $\Gamma_B' \equiv \sigma_G^2/\Omega_{2B}$ = ratio of intensity of the independent gaussian component, σ_G^2, of the input interference (including "front-end" receiver noise) to the intensity, Ω_{2B}, of the impulsive component, in the large-amplitude approximation pdf;

(3). Ω_{2B} = the intensity of the above-mentioned impulsive ("Class A-form") component;

(4). $$\hat{A}_\alpha = \frac{2\Gamma(1-\alpha/2)}{2^{3\alpha/2}\Gamma(1+\alpha/2)G_B^\alpha} A_B \left\langle\left(\frac{\hat{B}_{oB}}{\sqrt{2\Omega_{2B}(1+\Gamma_B')}}\right)^\alpha\right\rangle ,$$

(5.11c), an "effective" Impulsive Index, proportional to the Impulsive Index A_B, which depends on the generic parameter α;

(5). $\alpha = (\frac{2-\mu}{\gamma})_{surf};(\frac{3-\mu}{\gamma})_{vol}$ = spatial density-propagation parameter, cf. (4.24h); relating the exponents μ, γ of the range power-law dependencies of source density and propagation law, respectively, cf. (4.24f,g);

(6). N_I = scaling factor which insures that P_{1-B-I}, w_{1-B-I} are properly joined to P_{1-B-II}, w_{1-B-II};

(7). $z_{oB} = z_{oB}/\sqrt{\Omega_{2B}(1+\Gamma_B')}$: the (normalized) "bend-over" point, at which the two approximating forms of pdf (and PD, for $|z| \geq 0$) are joined, according to the procedures discussed in Sec. (5.2)ff. This is an empirically determined point, cf. Fig. 5.5.

As before, in the case of Class B envelope distributions, we have the following (same) sets of global and generic model parameters:

$$\text{Class B:} \quad \begin{array}{ll} \underline{\text{Global:}} & A_\alpha (\text{or } \hat{A}_\alpha), \alpha, A_B, \Gamma'_B, \Omega_{2B}, N_I \,; \\ \underline{\text{Generic:}} & A_B, \alpha, \sigma_G^2, \langle \hat{B}_{oB}^\alpha \rangle, \langle \hat{B}_{oB}^2 \rangle, N_I \end{array} \Bigg\} \,. \tag{2.8}$$

It is important to stress the fact that (except for z_{oB}) all model parameters are physically structured and hence are canonical in form: they are not ad hoc quantities. [We remark, also, that N_I is not fully dependent on the other generic parameters and is independent of z_{oB}. Thus, N_I may be regarded as generic.] We include two methods, one approximate and one exact [cf. Secs. 5.2-1,2] for obtaining the model parameters.

Finally, although we have no readily obtainable direct experimental comparisons vis-à-vis theory for the instantaneous amplitude, z, or its magnitude, $|z|$, we have inferentially, demonstrated excellent agreement between theory and experiment on the basis of the much more easily obtainable envelope data [cf. Figs. (2.1)-(2.8), Middleton, [1976]]. Since the models are basically the same, we can accordingly regard the experimental agreements, canonically achieved over a variety of disparate (Class B) sources, as confirming the adequacy of our model, also canonically. Because Class B models contain the α-parameter they are also capable of providing more structural information regarding these interference mechanisms, viz., average source densities (in space) for given power laws of propagation. [This point is discussed more fully, p.82, Middleton [1976].] Again, Hall-type models are deriveable from the B-I class of approximations [$|z_o|<|z_{oB}|$), when the additive gaussian component is negligible, cf. Section (5.3) below. Accordingly, with this Report we have accomplished Item 1 of "Next Steps" (cf. Sec. 2.7-I, Middleton, [(1976), and with current efforts (FY'78) we are underway on Item 2: first-order statistics of Class C interference, for various combinations of the Class A and Class B components which can comprise a typical EM interference environment. These results, in turn, are to be employed in the evaluation of the performance of optimum and suboptimum receivers in such environments, along the lines already developed by Spaulding and Middleton [1975, 1977].

Acknowledgments:

The author wishes to thank Dr. A.D. Spaulding of the National Telecommunications and Information Administration (ITS-Boulder) for his constructive comments and criticisms, and for his supervision of the numerical calculations for the Figures.

PART II. ANALYSIS

3. Introduction to the Analysis:

Part II of this Report, as in Ref. 11 earlier, is devoted to the derivation of the first-order instantaneous amplitude statistics $w_1(X)$, $P_1(X \geq X_0)$, and associated moments $\langle |X|^{\beta} \rangle$, primarily for our statistical-physical model of Class B interference. The analysis parallells much of Sections 2-7 (Part II) of Ref. 11, starting with the basic poisson model which describes these broad classes of interference. We are able here to take advantage of our previous results for the envelope statistics for Class A, B, and C noise generally,* and Class B in particular, using the methods developed in Sec. 2 (Part II), Ref. 11, to obtain first the needed characteristic functions and their approximating forms. Our principal attention is then directed to the Class B case. [For a full account of the physical reasoning underlying the approximations, see Sec. 2 (II), Ref. 11.]

The material of Part II here is organized as follows: in Section 4 below we obtain the general (integral) expressions for the desired PD's and pdf's of the instantaneous amplitudes, X, for Class A, B, and C interference.* Then, in Section 5 we specialize these results to Class B noise and evaluate the various integral forms involved, for $P_1(X \geq X_0)_B$. Corresponding results for the pdf, $w_1(X)_B$, are derived in Section 6. Section 7 presents an evaluation of the general first-order moments $\langle |X|^{\beta} \rangle$, $\beta \geq 0$; and offers remarks on the global and generic parameters of the model, along the lines of Sec. 6 (II), Ref. 11. Appendix A concludes the analysis with the derivation of the "joining" conditions on the PD's.

4. Integral Forms: $P_1(X \geq X_0)$, $w_1(X)$, Class A, B, C Interference:

Our starting point for the present analysis is the basic poisson model [Eq. (2.1), Ref. 11] for the first-order characteristic function of the instantaneous amplitude, X, of the received interference process (at the input to the IF stage of the typical narrow-band receiver), viz:

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* Class C models are to be treated in detail in a subsequent Report in this series (e.g., Part IV, 1978).

$$F_1(i\xi)_X = \exp\left\{\int_{\Lambda,\epsilon} \rho(\underset{\sim}{\lambda},\hat{\epsilon}) \left\langle \left(e^{i\xi U(t;\underset{\sim}{\lambda},\hat{\epsilon},\underset{\sim}{\theta}')} - 1\right)\right\rangle_{\underset{\sim}{\theta}'} d\underset{\sim}{\lambda}\, d\hat{\epsilon}\right\} . \tag{4.1}$$

Here $\hat{\epsilon}$ is an epoch indicating, vis-à-vis the receiver's time t, when a typical source may emit. [The basic model here, as before, consists of an infinite number of potentially interfering sources in a source domain, Λ, and while the basic waveforms emitted all have the same form, U, their scale, duration, frequency, etc. may be randomly distributed. The fundamental postulate of this basic interference model is that: the locations in space, and emission epochs, in time, are independent, and hence poisson distributed, cf. (4.1)] The $\underset{\sim}{\lambda} = (\lambda,\theta,\phi)$ are coördinates, or a vector magnitude, appropriate to the geometry of the source field, in Λ, and of the receiver, with $d\underset{\sim}{\lambda}$ (= $d\lambda d\phi$) for a surface distribution, and $d\underset{\sim}{\lambda}$ (= $d\lambda d\theta d\phi$) for a volume element. The quantity $\rho(\underset{\sim}{\lambda},\hat{\epsilon})$ is the "process density" of this point space-time poisson interference process, e.g., $\rho \geq 0$ ($\sim$ probability density). The average $\langle\ \rangle_{\underset{\sim}{\theta}}$ denotes a statistical average over various random parameters which may be pertinent to the particular model, e.g., doppler, signal duration, amplitude, etc. Finally, $U(t;\underset{\sim}{\lambda},\ldots)$ is the typical waveform,* in the receiver, following the (linear) aperture x RF x IF stages (i.e., ARI stages) of our narrow-band receiver, and X(t) is the resultant receiver process, given by

$$X(t) = \int_{\underset{\sim}{Z}=\underset{\sim}{\Lambda}\times\underset{\sim}{\theta}} U(t|\underset{\sim}{Z})\,dN(\underset{\sim}{Z}) \quad , \tag{4.2}$$

where the {dN} form a (usually zero-mean) poisson process (in $\underset{\sim}{\Lambda}\times\underset{\sim}{\theta}$), i.e., in $(\underset{\sim}{\lambda}\times\underset{\sim}{\theta})$, such that (4.1) is the first-order characteristic function.

The desired PD's and pdf's (probability distributions and probability density functions) are formally

$$\underline{\text{PD:}}\quad P_1(X\geq X_0) = \int_{X_0}^{\infty} w_1(X)\,dX \;; \qquad \underline{\text{pdf:}}\quad w_1(X) = \int_{-\infty}^{\infty} e^{-i\xi X} F_1(i\xi)_X \frac{d\xi}{2\pi} \quad . \tag{4.3}$$

We can rewrite the expression for P_1 in the following form, more convenient for evaluation:

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* A further generalization is possible when the waveform itself (U) is regarded as random: the averages $\langle\ \rangle_{\underset{\sim}{\theta}'}$, become $\langle\ \rangle_{\underset{\sim}{\theta}',U}$, cf. (4.1), etc.

$$P_1(X\geq X_o) = \lim_{X_1\to\infty} P_1(X_1>X\geq X_o) = \lim_{X_1\to\infty} \int_{X_o}^{X_1} w_1(X)dX \ . \tag{4.3a}$$

Using $w_1(X)$, (4.3), in this gives

$$P_1(X_1>X\geq X_o) = \int_{-\infty}^{\infty} F_1(i\xi)_X \frac{d\xi}{2\pi} \int_{X_o}^{X_1} e^{-i\xi X}dX$$

$$= \int_{-\infty}^{\infty} F_1(i\xi)_X \left[\frac{e^{-i\xi X_o}-e^{-i\xi X_1}}{i\xi}\right] \frac{d\xi}{2\pi} \ , \tag{4.4}$$

and since $X(t)$ is an instantaneous amplitude of a narrow-band process, X is a zero mean ($\langle X\rangle = 0$), symmetrical process about $X = 0$, e.g., $w_1(X) = w_1(-X)$ and $F_1(i\xi)_X = F_1(-i\xi)_X$, i.e., the c.f. is likewise even in ξ. Equation (4.4) accordingly can be rewritten as

$$P_1(X_1>X\geq X_o) = \frac{1}{\pi}\int_{0-}^{\infty} \left[\frac{\sin \xi X_1-\sin \xi X_o}{\xi}\right] F_1(i\xi)_X d\xi, \tag{4.4a}$$

$$P_1(|X_1|>|X|\geq|X_o|) = \frac{2}{\pi}\int_{0-}^{\infty}\left[\frac{\sin \xi|X_1|-\sin \xi|X_o|}{\xi}\right] F_1(i\xi)_X d\xi \ , \tag{4.46}$$

so that

$$P_1(X\geq X_o) = \frac{1}{2}\left[1- \frac{2}{\pi}\int_{0-}^{\infty} \frac{\sin \xi X_o}{\xi} F_1(i\xi)_X d\xi\right] \ , \tag{4.5}$$

since (heuristically)

$$\lim_{X_1\to\infty} \frac{\sin \xi X_1}{\xi} = \frac{\pi}{2}\,\delta(\xi-0) \quad ; \quad F_1(0)_X = 1 \ \text{(always)} \ , \tag{4.5a}$$

and $\int_{0-}^{\infty} \delta(y-0)F(y)dy = F(0)$. [Note, as expected, that if $X_o = 0$, $P_1(X\geq 0) = 1/2$, while $X_o \to -\infty$, $P_1(X\geq -\infty) = 1/2-(-1/2) = 1$, from (4.5), also as required.]

Similarly, for the more special case $P_1(|X|\geq|X_o|)$, we have directly here $[w_1(X) = w_1(-X)]$:

$$P_1(|X|\geq|X_0|\geq 0) = 2\int_{|X_0|(\geq 0)}^{\infty} w_1(X)dX = 1 - \frac{2}{\pi}\int_{0-}^{\pi} \frac{\sin\xi|X_0|}{\xi} F_1(i\xi)_X d\xi. \tag{4.5b}$$

Finally, for these symmetrical distributions, we obtain from (4.3)

$$w_1(X) = \frac{1}{\pi}\int_{0-}^{\infty} \cos \xi X \; F_1(i\xi)_X d\xi = -\left.\frac{dP_1}{dX_0}\right|_{X_0\to X} \quad (= w_1(-X)) \; . \tag{4.6}$$

Both (4.5), (4.5b), and (4.6) will be employed here in the subsequent analysis. Our principal task, accordingly, is to apply (4.1) to these relations, to obtain analytically tractable expressions for the various PD's and pdf's of X, as in our earlier studies [Middleton, 1974, 1976].

Our next step is to use (2.10), Ref. 11 for the basic waveform, U, in the receiver viz:

$$U = U_{\text{narrow-band}} = B_0(t,\underset{\sim}{\lambda}|\hat{\epsilon},\underset{\sim}{\theta})\cos \mu_d\Psi(t,\underset{\sim}{\lambda}|\hat{\epsilon},\underset{\sim}{\theta}) \; ; \; \mu_d = 1+\epsilon_d \; , \tag{4.7}$$

where B_0 (≥ 0) is an envelope (cf. (4.10a)) and Ψ is a phase, with the form

$$\Psi \equiv \omega_0(t-\lambda-\hat{\epsilon})-\mu_d^{-1}[\Phi_S(t-\lambda-\hat{\epsilon},\underset{\sim}{\theta})+\phi_T(\underset{\sim}{\lambda},f_0)+\phi_R(\underset{\sim}{\lambda},f_0)] \; . \tag{4.7a}$$

Here Φ_S, ϕ_T, ϕ_R are respectively the typical source phase, and the phase angles of the (complex) beam patterns of the source (T) and receiver (R), [cf. Sec. 2.5, (II), Ref. 11]. The quantity ϵ_d ($=\mu_d-1$) is the sum of the relative dopplers between a source and the receiver, and is always small $0(10^{-5}$ or less) in our applications, viz. $\epsilon_d = 2v/c = 0(10^{-6})$ for $v=10^5$ mph, so that B_0 is independent of ϵ_d. Inserting (4.7) into (4.1) and applying the narrow-band conditions, e.g., following the procedure on pp. 44, 45, Ref. 11 (or pp. 24, 25, Ref. 9), we obtain for the c.f.

$$F_1(i\xi)_X = \exp\left\{\int_{\Lambda,\hat{\epsilon}} \rho(\underset{\sim}{\lambda},\hat{\epsilon}) \left\langle J_0(\xi B_0[t;\underset{\sim}{\lambda}|\underset{\sim}{\hat{\epsilon}},\underset{\sim}{\theta}])-1\right\rangle_{\underset{\sim}{\theta}} d\underset{\sim}{\lambda} d\hat{\epsilon}\right\} , \tag{4.8}$$

which is still essentially exact. (Here J_0 is a Bessel function of the first-kind, order zero, as before.)

Our next step is to follow the analysis [pp. 46-50, Ref. 11], making the following not very restrictive assumptions:

(i). average no. of emissions per unit domain ($d\Lambda$) and per interval $d\hat{\epsilon}$ in (0,T) are independent [(2.25)II, Ref. 11];

(ii). "local stationarity": no changes in average source numbers and emission properties during the observation period (0,T), and the emission epoch probability (density) $w_1(\hat{\epsilon})$ is uniform in all intervals;

(iii). idealized "steady-state" condition ($T\to\infty$) holds.

The result is that (4.8) is now reduced to the basic form

$$\boxed{F_1(i\xi)_X = \exp\left\{A_\infty\left\langle\int_0^{Z_0=T_s/\bar{T}_s}[J_0(\xi\hat{B}_0)-1]dz\right\rangle_{\underset{\sim}{\lambda},T_s,\underset{\sim}{\theta}'}\right\},} \qquad (4.9)$$

where specifically,

$$\hat{B}_0 = B_0(z\bar{T},\underset{\sim}{\lambda};\underset{\sim}{\theta}') = |\mathcal{A}_R(\underset{\sim}{\lambda},f_0)\mathcal{A}_T(\underset{\sim}{\lambda},f_0)|A_0 e_{0\gamma}u_0(z)g(\underset{\sim}{\lambda}) \qquad (4.10a)$$

A_0 = (peak) amplitude of the received envelope (at output of the IF stage) (4.10b)

$e_{0\gamma}$ = a limiting voltage setting (in suitable dimensions), at which the receiver will respond to a test signal, above receiver noise, at the output of the IF [cf. (2.34), II, Ref. 11]; (4.10c)

$u_0(z)$ = normalized envelope wave-form at IF output; ($z=t/\bar{T}_s$);
= 0, $z > T_s/\bar{T}_s$, $z < 0$; (4.10d)

T_s = duration of typical signal from an interfering source (4.10e)

$\bar{T}_s$ = mean duration of typical signal from an interfering source (4.10f)

$\underset{\sim}{\theta}'$ = other random parameters of interference waveform (other than $\underset{\sim}{\lambda}$, T_s); (4.10g)

$g(\underset{\sim}{\lambda})$ = a geometrical factor, which describes the propagation law, from source to receiver (in their mutual-far fields); (4.10h)

$\mathcal{a}_T, \mathcal{a}_R$=(complex) beam patterns of typical source and the receiver. (4.10i)

[Note that (4.9) is formally identical to exp [Eq. (2.38), Middleton, 1976], with ξ replacing r therin.]

The quantity A_∞ is the <u>Impulsive Index</u> (as defined in (2.37), (2.38)II (Middleton, 1976)). As noted in our earlier studies [Middleton, 1972b, 1973, 1974, 1976], the Impulsive Index is a measure of the temporal overlap or density, at any instant, of the superposed interference waveforms <u>at the receiver's IF output.</u> It is a key parameter of the interference model because it critically influences the form of the pdf's and PD's of the interference, as observed at the output of the initial (linear) stages of our typical narrow-band receiver. Then, for small values of A_∞ the statistics of the resultant ouput wave are dominated by the overlapping of comparatively few, deterministic waveforms, of different amplitudes and durations. The resulting interference has then an "impulsive", structured character. For increasingly large values of A_∞ this resultant approaches a normal (or gaussian) process as one would expect from the Central Limit Theorem (CLT) [Middleton, 1960, Sec. 7.7], cf. Eq. 4.19 following.

We are now ready to use the procedures and results of Sections 2.3, 2.4 (II), [Middleton, 1976] to obtain expressions for $F_1(i\xi)_X$ when the interference belongs to Class A, B, or C types. Since r (therein) $\rightarrow$ ξ (here), we have at once, for the general Class C case:

$$\left\{\begin{aligned} F_1(i\xi)_C &= \exp\left\{A_A\left\langle\int_0^{z_0(=T_s/\bar{T}_s)<\infty}[J_0(\xi\hat{B}_{oA})-1]dz\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'}\right. \\ &\qquad\left. +A_B\int_0^\infty\left\langle[J_0(\xi\hat{B}_{oB})-1]\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'}dz\right\} \qquad (4.11) \\ &= F_1(i\xi)_A \cdot F_1(i\xi)_B \,, \qquad (4.11a)\end{aligned}\right.$$

where the impulsive Indexes A_A, A_B are specifically

$$A_A = \bar{\nu}_\infty\bar{T}_{sA} \;;\; A_B = \bar{\nu}_\infty\bar{T}_{sB} \;; \text{ and}$$

$$\langle\;\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'} \equiv \int_0^\infty(\;)w(z_0)dz_0\int_{\Lambda,\underset{\sim}{\theta}'}w_1(\underset{\sim}{\theta}')\frac{\rho(\underset{\sim}{\lambda})}{A_\Lambda}(\;)d\underset{\sim}{\lambda}d\underset{\sim}{\theta}' \qquad (4.12)$$

in which $A_\Lambda \equiv \int_\Lambda\rho(\underset{\sim}{\lambda})d\underset{\sim}{\lambda}$ = av. no. of emitting sources in Λ. Note that $\hat{B}_{oA} \neq \hat{B}_{oB}$, cf. (4.10a), through the waveform $u_0(z)$, e.g. $u_0(z)_B \neq 0$ $(0 \leq z < \infty)$, whereas $u_0(z)_A = 0$ outside $(0, \bar{z}_0 < \infty)$, cf. Fig. 2.1(II), Middleton, 1976] p. 53.

In general, there is always an accompanying gaussian background, which arises from a number of mechanisms:

(i). as <u>system noise</u> in the receiver;
(ii). as <u>external interference</u>, which is the resultant of many independent sources, none of which is dominating vis-à-vis the others (so that the CLT applies);
(iii). as a mixture of (i) and (ii). (4.13)

This component of the interference is also independent of the "impulsive" component. Accordingly, we may write for the c.f. of the (sum of) these components:

$$F_1(i\xi)_{P+G} = F_1(i\xi)_P \cdot F_1(i\xi)_G \ , \tag{4.14}$$

for P (= poisson) and G (= gauss), where specifically

$$F_1(i\xi)_G = e^{-\sigma_G^2\xi^2/2} \ ; \quad \sigma_G^2 = \sigma_E^2 + \sigma_R^2 \ , \tag{4.15}$$

where σ_E^2, σ_R^2 are respectively the variances of the external and the receiver (gauss) components.

Applying (4.14), (4.15) to (4.11), and noting that for Class A interference ($A_A >> A_B$); for Class B interference ($A_B >> A_A$), while for the general Class C "mixture", ($A_A \sim A_B$), we have specifically

$$F_1(i\xi)_{X:C+G} = F_1(i\xi)_C \cdot e^{-\xi^2\sigma_G^2/2} \ , \quad [\text{cf. Eq. (4.11)}]; \tag{4.16}$$

Class A (Amplitude) Interference+Gauss:

$$F_1(i\xi)_{X:A+G} = \exp\left\{-\xi^2\sigma_G^2/2 - A_A + A_A\left\langle\int_0^{z_0(<\infty)} J_0(\xi\hat{B}_{0A})dz\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'}\right\}; \tag{4.17}$$

Class B (Amplitude) Interference+Gauss:

$$\boxed{F_1(i\xi)_{X:B+G} = \exp\left\{-\xi^2\sigma_G^2/2 + A_B\int_0^\infty \left\langle[J_0(\xi\hat{B}_{0B})-1]\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'} dz\right\} .} \tag{4.18}$$

It is Eq. (4.18) with which we shall be primarily considered here.

Parallelling the analysis of Sec. 2.4(II), [Middleton, 1976] we readily find that for <u>large</u> Impulsive Indexes the c.f.'s are asymptotically gaussian, e.g.

$$F_1(i\xi)_{C+G} \doteq e^{-\sigma_0^2\xi^2/2}[1+O(\xi^4)] \ , \tag{4.19}$$

where

$$\sigma_0^2 = \sigma_G^2 + \frac{A_A}{2}\left\langle\int_0^{z_0(<\infty)} \hat{B}_{0A}^2 dz\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'} + \frac{A_B}{2}\left\langle\int_0^{\infty} \hat{B}_{0B}^2 dz\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'} \quad . \tag{4.19a}$$

The specific structure of the "correction" (or Edgeworth series) terms in (4.19) follows at once from (2.53), (2.54), [Middleton, 1976]. The associated pdf and P.D. for (4.19) are obtained directly from (4.3)-(4.6) and are the familiar forms

$$w_1(X)_{C+G} \simeq \frac{e^{-X^2/2\sigma_0^2}}{\sqrt{2\pi\sigma_0^2}} \; ; \quad \left.\begin{array}{l} P_1(X\geq X_0) \simeq 1/2[1-\Theta(X_0/\sigma_0\sqrt{2})] \; ; \\ P_1(|X|\geq|X_0|) \cong 1-\Theta(|X_0|/\sigma_0\sqrt{2}) \; , \end{array}\right\} \; , \tag{4.20}$$

where $\Theta(z) \equiv (2/\sqrt{\pi})\int_0^z e^{-t^2} dt$ is the familiar (tabulated) <u>error integral.</u> For Class A, Class B noise these relations simplify at once on setting

<u>Class A:</u>

$$\sigma_0^2 \to \sigma_{0A}^2 \to \sigma_G^2 + \frac{A_A}{2}\left\langle\int_0^{z_0(<\infty)} \hat{B}_{0A}^2 dz\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'} \; ;$$

<u>Class B:</u>

$$\sigma_0^2 \to \sigma_{0B}^2 = \sigma_G^2 + \frac{A_B}{2}\left\langle\int_0^{\infty} \hat{B}_{0B}^2 dz\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'} \quad . \tag{4.21}$$

There remains the final reduction of the c.f.'s to give us the desired tractable analytic forms, suitable for inversion, to obtain the pdf's and P.D.'s involved. At this point we take advantage of the formal identity of our results (4.11), (4.16)-(4.18), with the expressions in Secs. (2.3) -(2.5)II, [Middleton, 1976], where now $r \to \xi$. The same physical and analytical arguments for the reduction [Sec. 2.5(II) therein] apply here, so that we can write down at once the various desired (approximate) forms for the c.f.'s. These are:

$$F_1(i\xi)_{A+G} \doteq \exp\left\{-\sigma_G^2\xi^2/2-A_A+A_A e^{-\xi^2\langle\hat{B}_{oA}^2\rangle/4}\right\}$$

$$\doteq e^{-A_A}\sum_{m=0}^{\infty}\frac{A_A^m}{m!}\,e^{-[m\langle\hat{B}_{oA}^2\rangle/2+\sigma_G^2]\xi^2/2}\ , \tag{4.22}$$

with correction terms given specifically by (2.78)II, [Middleton, 1976], $r \to \xi$. The various pdf's, PD's, moments, etc. for the instantaneous amplitude X have been determined and discussed in detail in "Part I" [Middleton, 1974] for this Class A interference, and so will not be considered further here.

On the other hand, for the (first-order) statistics of the Class B noise process, X, we must consider the dual characteristic functions*

$$\boxed{\begin{aligned} F_1(i\xi)_{B+G-I} &\doteq e^{-b_{1\alpha}A_B|\xi|^{\alpha}-\Delta\sigma_G^2\xi^2/2} && (4.23a)\\ & && (0<\alpha<2)\\ F_1(i\xi)_{B+G-II} &\doteq e^{-A_B}\exp[A_B e^{-b_{2\alpha}\xi^2/2}-\sigma_G^2\xi^2/2]\ , && (4.23b)\end{aligned}}$$

where the subscript -I indicates the c.f. appropriate to the range $(0\le|X|,|X_o|\le|X_B|)$ of amplitude values, while the subscript -II denotes the c.f. associated with the amplitude range $(|X_B|\le|X_o|\le|X|)$, cf. (3.10a,b), [Middleton, 1976]. The precise definition of the boundary point, $|X_B|$, will be given presently, in Sec. 5. The parameters of (4.23a,b) are

$$b_{1\alpha} \equiv \frac{\Gamma(1-\alpha/2)}{2^{\alpha-1}\Gamma(1+\alpha/2)}\langle G_{oB}^{\alpha}\rangle/\lambda_{max}^{\alpha\gamma} = \frac{\Gamma(1-\alpha/2)}{2^{\alpha/2-1}\Gamma(1+\alpha/2)}\left\langle\left(\frac{\hat{B}_{oB}}{\sqrt{2}}\right)^{\alpha}\right\rangle\ (>0)\ ; \tag{4.24a}$$

---

* Note that we have $\xi\to|\xi|$ here, since $J_0(\xi\hat{B}_{oB}) = J_0(|\xi|\hat{B}_{oB})$, and since the c.f. must be an even function of ξ, in view of the symmetry of the pdf about X = 0.

$$b_{2\alpha} \equiv \left(\frac{4-\alpha}{2-\alpha}\right)\left\langle G_{oB}^{2}\right\rangle/\lambda_{max}^{2} = \left(\frac{4-\alpha}{2-\alpha}\right)\left\langle \hat{B}_{oB}^{2}\right\rangle/2 \quad (>0) \; ; \tag{4.24b}$$

$$G_{oB}^{\alpha} = e_{o\gamma}^{(B)^{\alpha}} A_o^{\alpha} |\mathcal{A}_{RT}|^{\alpha} g_{S,V}^{\alpha} (4\pi c)^{-\alpha\gamma} \int_0^{\infty} u_o(z)_B^{\alpha} dz \; , \quad (0 \leq \alpha \leq 2) \; ; \tag{4.24c}$$

$$\Delta\sigma_G^2 \equiv \sigma_G^2 + b_{2\alpha} A_B = \sigma_R^2 + \sigma_E^2 + b_{2\alpha} A_B \quad (>0) . \tag{4.24d}$$

The key analytic differences between (4.23a) and (2.90)II, [Middleton, 1976] is the term $|\xi|^{\alpha}$: the absolute value of $|\xi|$ is required in this approximation, to conform to the symmetry condition imposed on $F_1(i\xi)_X$, cf. (4.4) et seq., which is in turn demanded by the narrow-band nature of the instantaneous amplitude X, which is symmetrical about $\langle X\rangle = \bar{X} = 0$. [From the purely analytic viewpoint, the exact expressions (4.11), (4.17), (4.18) are even in ξ: hence the above approximate forms (4.22), (4.23) must be likewise.]

Finally, we must specify the remaining parameters: $\lambda_{max}, \gamma, \alpha$. These are:

λ_{max} = limiting range ($\div c$), depending on $e_{o\gamma}$ [cf. (4.10c)], of the receiver; (4.24e)

γ = exponent of the propagation law: $g(\underset{\sim}{\lambda}) = [g_S(\phi, g_V(\theta,\phi))]/(4\pi c\lambda)^{\gamma}$, cf. (4.10h) above, and [(2.61), Middleton, 1976], e.g. $g \sim 1/\lambda^{\gamma}$, $0 < \gamma$; (4.24f)

μ = exponent of the density law with range ($\sim c\lambda$), e.g. $\sigma_{S,V}(\lambda) = 1/\lambda^{\mu}$, $0 < \mu$; cf. [2.63), Middleton, 1976]; (4.24g)

$$\alpha \equiv \left.\frac{2-\mu}{\gamma}\right|_{surface} ; \quad \left.\frac{3-\mu}{\gamma}\right|_{volume} : \tag{4.24h}$$

<u>spatial density-propagation parameter</u>; [In general, see Sec. 2.7, and Eq. (2.82), Middleton, 1976]. (For our present applications, the range of values of α is: <u>$0 < \alpha < 2$</u>.) (4.24i)

$g_{s,v}$ = angular geometrical factors in the propagation law, usually taken to be unity; cf. discussion, Sec. 2.5.1 [Middleton, 1976]. (4.24j)

As we have noted earlier [p. 82, Middleton, 1976], it is important to distinguish the qualitative differences between Class A and Class B interference, in their amplitude statistics (as well as earlier, in their envelope properties):

(i). Unlike Class A interference, the (first-order) statistics of Class B noise are clearly sensitive to the combined effects of source-distribution law (μ) and propagation law (γ), through the generic parameter α, cf. (4.24h).

(ii). Consequently, the sensitivity to α is a receiver bandwidth phenomenon, which can be (in principle) removed by suitably broadening receiver bandwidth Δf_R vis-à-vis Δf_N, or heightened by suitably narrowing Δf_R vis-à-vis Δf_N. In fact, Class B operation is often more desireable from a measurement viewpoint, as it increases the number of descriptive parameters of the interference source from 3 [Class A, cf. (4.22)] to 6 [Class B, cf. (4.23)], and thus provides the potential for a more detailed description of source structure and characteristics, via (4.24).

(iii). The Class A and B noise parameters are the same (as we would expect on reflection) for both the amplitude (here) and envelope statistics [Middleton, 1976]: (4.17), (4.22) and (4.18), (4.23) here.

5. Probability Distributions $P_1(X \geq X_0)_{B+G}$; $0 < \alpha < 2$:

We proceed here to use the desired approximations (4.23a,b) to the characteristic function $F_1(i\xi)_{B+G}$ in (4.5,5b), to obtain the corresponding (approximate) forms for $P_1(X \geq X_0)_{B+G}$.

First, we remark that just as in the earlier analysis for the envelope statistics of Class B noise the approximating c.f.'s here [cf. (4.23a,b)] yield pdf's (and PD's) which are not properly scaled. In fact, as we shall note below, the c.f. F_{1-I}, (4.23a), yields a pdf $w_1(X)_{B-I}$, which requires a change of scale for the argument X (or $z \equiv X/\langle X^2\rangle^{1/2} = X/\sqrt{\Omega_{2B}(1+\Gamma_B')}$) and $\therefore X_o$ (or z_o) in the associated PD. Moreover, if $w_1(X)_{B-I}$ is used over the whole range of X: $(-\infty < X < \infty)$, then $\langle X^2\rangle_I \to \infty$. [On the other hand, the c.f. F_{1-II}, (4.23b), yields a pdf, $w_1(X)_{B-II}$, which gives a finite second moment over $(-\infty,\infty)$, e.g., $\langle \dot{z}^2\rangle_{II} = 4G_B^2(\neq 1)$, cf. Eq. (7.4) below, but a moment which is not suitably normalized until we divide by $4G_B^2$.]

With this in mind let us next introduce the following normalizations:

$$z \equiv X/\sqrt{\Omega_{2B}(1+\Gamma_B')} \;;\quad z_o \equiv X_o/\sqrt{\Omega_{2B}(1+\Gamma_B')} \;;\quad z_{oB} = X_{oB}/\sqrt{\Omega_{2B}(1+\Gamma_B')} \; (>0) \tag{5.1}$$

with

$$\Omega_{2B} \equiv A_B\langle \hat{B}_{oB}^2\rangle/2 \quad ; \quad \Gamma_B' \equiv \sigma_G^2/\Omega_{2B} \tag{5.1a}$$

$$\therefore \quad \Omega_{2B}(1+\Gamma_B') = \Omega_{2B}+\sigma_G^2 = \langle X^2\rangle_{total} \quad , \tag{5.1b}$$

(as before, cf. (3.2a), [Middleton, 1976]). Letting $\hat{a}$ be a normalizing factor

$$\hat{a} \equiv [\Omega_{2B}(1+\Gamma_B')]^{-1/2} \;, \tag{5.2}$$

analogous to $a \equiv \{2\Omega_{2B}(1+\Gamma_B')\}^{-1/2}$ $(= 2^{-1/2}\hat{a})$, cf. (3.3), [Middleton, 1976], we can write

$$\therefore \; \xi = \hat{a}\lambda \quad ; \quad z = \hat{a}X \quad ; \quad z_o = \hat{a}X_o \;;\; z_{oB} = \hat{a}X_B \;,\; \text{cf. (4.23)}, \tag{5.3}$$

where λ, z, z_o are now dimensionless quantities. Then the approximating c.f.'s (4.23) become

$$F_1(i\xi)_{B-I} \doteq e^{-b_{1\alpha}A_B\hat{a}^\alpha|\lambda|^\alpha - \Delta\sigma_G^2\hat{a}^2\lambda^2/2} \equiv \hat{F}_1(ia\lambda)_{B-I} \;;\; I:(|z|,|z_o|<|z_{oB}|), \tag{5.4a}$$

$$F_1(i\xi)_{B-II} \doteq e^{-A_B} \exp[-\sigma_G^2\hat{a}^2\lambda^2/2+A_B e^{-b_{2\alpha}\hat{a}^2\lambda^2/2}] \equiv \hat{F}_1(ia\lambda)_{B-II}. \tag{5.4b}$$

$$\text{II: } |z|,|z_o|>|z_{oB}|.$$

The associated PD's are accordingly from (4.5) and (4.5b), cf. Appendix A:

$$\therefore P_1(z\geq z_o)=P_1(z\geq z_o)_{B-I} \quad |z_{oB}|<z_o<|z_{oB}| \left.\begin{matrix} \\ \text{B-II: } z_o<-|z_{oB}| \\ z_o>|z_{oB}| \end{matrix}\right\}$$

$$\cong \frac{1}{2}[1-\frac{2}{\pi}\int_{0-}^{\infty} \frac{\sin z_o\lambda}{\lambda} \hat{F}_1(ia\lambda)_{B-I} \quad |z_{oB}|<z_o<z_{oB} \left.\begin{matrix} \\ \text{B-II } z_o<-|z_{oB}|, z_o>z_{oB}(>0)\end{matrix}\right\} \tag{5.5a}$$

$$\therefore P_1(|z|\geq|z_o|) = P_1(|z|\geq|z_o|)_{B-I} \quad 0\leq|z_o|<z_{oB} \left.\begin{matrix} \\ \text{B-II } |z_o|>z_{oB}\end{matrix}\right\}$$

$$\cong 1-\frac{2}{\pi}\int_{0-}^{\infty} \frac{\sin|z_o|\lambda}{\lambda} \hat{F}_1(ia\lambda)_{B-I} \quad 0\leq|z_o|\leq z_{oB} \left.\begin{matrix} \\ \text{B-II } |z_{oB}|>z_{oB}\end{matrix}\right\} d\lambda \tag{5.5b}$$

for these symmetrical distributions, subject to the "joining" condition $P_1(|z|\geq|z_o|=|z_{oB}|)_{B-I} = P_1(|z|\geq|z_{oB}|=|z_{oB}|)_{B-II}$, cf. (5.22a), (5.26a).

5.1 Evaluations of $P_{1-B-I,II}$:

To facilitate the integration we replace $\sin \lambda z_o$ by its equivalent Bessel-function form [Watson, 1944] viz:

$$\sin z_o\lambda = \sqrt{\frac{\pi z_o\lambda}{2}}\; J_{1/2}(z_o\lambda), \tag{5.6}$$

so that (5.5a,b), with the help of 5.4a,b), become explicitly

$$P_1(z\geq z_o)_{B-I} \simeq \frac{1}{2} - \sqrt{\frac{z_o}{2\pi}}\int_{0-}^{\infty} \lambda^{-1/2} J_{1/2}(z_o\lambda) e^{-b_{1\alpha}A_B\hat{a}^{\alpha}|\lambda|^{\alpha}-\Delta\sigma_G^2\hat{a}^2\lambda^2/2} d\lambda \tag{5.7a}$$

$$P_1(z\geq z_0)_{B-II} \simeq \frac{1}{2} - e^{-A_B}\sqrt{\frac{z_0}{2\pi}} \sum_{m=0}^{\infty} \frac{A_B^m}{m!} \int_{0-}^{\infty} \lambda^{-1/2} J_{1/2}(z_0\lambda) e^{-\hat{c}_{mB}^2\lambda^2/2} d\lambda, \quad (5.7b)$$

$$\left.\begin{aligned} \text{with:}\quad \hat{c}_{mB}^2 &\equiv (mb_{2\alpha}+\sigma_G^2)\hat{a}^2 = \left(\frac{mb_{2\alpha}}{\Omega_{2B}} + \Gamma_B'\right)/(1+\Gamma_B') = \\ \hat{A}_B &\equiv A_B\left(\frac{2-\alpha}{4-\alpha}\right); \quad = \left(\frac{m}{\hat{A}_B} + \Gamma_B'\right)/(1+\Gamma_B') \end{aligned}\right\} . \quad (5.7c)$$

Also, we have (from (5.5b))

$$P_1(|z|\geq|z_0|)_{B-I} \simeq 1 - \sqrt{\frac{2|z_0|}{\pi}} \int_{0-}^{\infty} \lambda^{1/2} J_{1/2}(|z_0|\lambda) e^{-b_{1\alpha}A_B\hat{a}^\alpha|\lambda|^\alpha - \Delta\sigma_G^2\hat{a}^2\lambda^2/2} d\lambda, \quad (5.8a)$$

$$P_1(|z|\geq|z_0|)_{B-II} \simeq 1 - e^{-A_B}\sqrt{\frac{2|z_0|}{\pi}} \sum_{m=0}^{\infty} \frac{A_B^m}{m!} \int_{0-}^{\infty} \lambda^{-1/2} J_{1/2}(|z_0|\lambda) e^{-\hat{c}_{mB}^2\lambda^2/2} d\lambda, \quad (5.8b)$$

with (5.7c), and where we have used (4.24b), (5.1)-(5.3) in defining the parameters $\hat{c}_{mB}^2$, $\hat{A}_B$.

Our principal task now is to evaluate (5.6)-(5.8). For this, we need the Hankel integral result [cf. (A.1-19), Middleton, 1960, for example]

$$\int_0^\infty J_\nu(az) z^{\mu-1} e^{-b^2z^2} dz = \frac{\Gamma\left(\frac{\nu+\mu}{2}\right)(a/2b)^\nu}{2b^\mu \Gamma(\nu+1)} \, {}_1F_1\left(\frac{\mu+\nu}{2};\nu+1;-a^2/4b^2\right) \quad (5.9)$$

$$\mathrm{Re}(\nu+\mu)>0;\ |\arg b| < \pi/4 .$$

Our first step in dealing with the B-I form (for $0 \leq |X| \leq |X_B|$) parallels the earlier analysis to obtain the analogous envelope statistics: $P_1(\mathcal{E} > \mathcal{E}_0)_{B-I} \equiv P_1(\hat{\mathcal{E}} > \hat{\mathcal{E}}_0)$, Eqs. (3.11a,b), [Middleton, 1976]. We first replace z_0 by $z_0 N_I$ in (5.7a), (5.8a), where $N_I(>0)$ is a scaling factor, which helps to insure that $\langle X^2\rangle_B = 1$. Next, we define a parameter G_B by

$$\hat{a}^2\Delta\sigma_G^2 \equiv 4G_B^2 \quad , \quad \text{or} \quad G_B^2 = \frac{1}{4}\left(\frac{\Gamma_B' + \frac{4-\alpha}{2-\alpha}}{1+\Gamma_B'}\right) , \quad (5.10)$$

(which is the same as the G_B, (3.12b), [Middleton, 1976]). The simplest useful form of result is obtained by expanding the exponent containing $|\lambda|^{\alpha}$ and then using (5.9). We obtain after a little manipulation

$$P_1(z \geq z_o)_{B-I} \equiv \hat{P}_1(\hat{z} \geq \hat{z}_o)_I \tag{5.11a}$$

$$\simeq \frac{1}{2} - \frac{\hat{z}_o}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{n\alpha+1}{2})}{n!} \hat{A}_{\alpha}^n \, {}_1F_1(\frac{n\alpha+1}{2}; \frac{3}{2}; -\hat{z}_o^2) \, ,$$

$$P_1(|z| \geq |z_o|)_{B-I} \equiv \hat{P}_1(|\hat{z}| \geq |\hat{z}_o|)_I \tag{5.11b}$$

$$\simeq 1 - \frac{2|\hat{z}_o|}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{n\alpha+1}{2})}{n!} \hat{A}_{\alpha}^n \, {}_1F_1(\frac{n\alpha+1}{2}; \frac{3}{2}; -|\hat{z}_o|^2) \, ,$$

where

$$\left.\begin{aligned} \hat{A}_{\alpha} &\equiv A_{\alpha}/2^{\alpha}G_B^{\alpha} = (2^{\alpha}b_{1\alpha}a^{\alpha}A_B)/2^{\alpha}G_B^{\alpha} = \frac{b_{1\alpha}2^{-\alpha/2}A_B}{[\Omega_{2B}(1+\Gamma_B')]^{\alpha/2}G_B^{\alpha}} \\ (A_{\alpha} &\equiv 2^{\alpha}b_{1\alpha}a^{\alpha}A_B); \; (a = 2^{-1/2}\hat{a}) \; ; = \frac{2\Gamma(1-\alpha/2)}{2^{3\alpha/2}G_B^{\alpha}\Gamma(1+\alpha/2)} A_B \left\langle \left(\frac{\hat{B}_{oB}}{\sqrt{\Omega_{2B}(1+\Gamma_B')}} \right)^{\alpha} \right\rangle \end{aligned}\right\} \tag{5.11c}$$

and where

$$\hat{z}_o \equiv (z_o N_I / 2\sqrt{2}\, G_B) \, . \tag{5.11d}$$

Here $\hat{A}_{\alpha}$ is again the <u>Effective Class B Impulsive Index</u>, which is unchanged from that quantity defined in (3.12a,b), [Middleton, 1976], and is proportional to the Impulsive Index, A_B, here. Like A_B, $\hat{A}_{\alpha}$ also depends on the spatially sensitive parameter α, and on the <u>relative</u> gauss component Γ_B', cf. (5.1a).

With the help of Kummer's transformation [cf. Middleton, 1960, Eq. A.1-17] we see that when $\hat{z}_o \to \pm\infty$,

$$ {}_1F_1\left(\frac{n\alpha+1}{2}, 3/2; -\hat{z}_o^2\right) \simeq \frac{\Gamma(3/2)}{\Gamma\left(\frac{2-n\alpha}{2}\right)} |\hat{z}_o|^{-n\alpha} \cdot \hat{z}_o^{-1}[1+0(|z_o|^{-1})] , \tag{5.12a} $$

so that the leading term in the series in (5.11a,b) become respectively

$$ \frac{1}{2}\hat{z}_o/|\hat{z}_o|]_{n=0} \to 1/2;-1/2 \text{ as } \hat{z}_o \to \pm\infty. \tag{5.12b} $$

Accordingly, we see, as expected, that P_1 takes the proper limits as $z_o \to \pm\infty, 0$:

$$ \left.\begin{aligned} P_1(z \geq z_o = -\infty)_{B-I} &= 1/2 - (-1/2) = 1 \\ P_1(z \geq \infty)_{B-I} &= 1/2 - 1/2 = 0 \\ P_1(z \geq 0)_{B-I} &= 1/2 \end{aligned}\right\} . \tag{5.13} $$

Similarly, we have at once $P_1(|\hat{z}| \geq 0)_{B-I} = 1$, $P_1(|\hat{z}| > \infty) = 0$, also, as expected. This shows that these expressions for the PD give the correct limiting forms, and are now scaled (N_I) to permit proper "joining" to P_{B-II}, so as to give $\langle z^2 \rangle = 1$ (or $\langle X^2 \rangle = \Omega_{2B} + \sigma_G^2 = \Omega_{2B}(1+\Gamma_B')$, as noted at the beginning of this section.

For large values of $\hat{z}_o$ we use, as before, [cf. (3.14), [Middleton, 1976]) the asymptotic expansion

$$ {}_1F_1(\alpha;\beta;-x) \simeq \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} x^{-\alpha}\left[1 + \frac{\alpha(\alpha-\beta+1)}{1!x} + \frac{\alpha(\alpha+1)(\alpha-\beta+1)(\alpha-\beta+2)}{2!x^2} + \ldots\right] , \tag{5.14} $$

to get for (5.11a), (5.11b)*:

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* For (5.11b), in (5.15) we replace $\hat{z}_o$ by $|\hat{z}_o|$ and multply by 2.

$$\hat{P}_1(z\geq z_o) \simeq \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\hat{A}_\alpha^n \Gamma(\frac{n\alpha+1}{2})}{\sqrt{\pi}\, n!\Gamma(\frac{2-n\alpha}{2})} |\hat{z}_o|^{-n\alpha}[1+\frac{(n\alpha+1)n\alpha}{4\hat{z}_o^2} + O(\hat{z}_o^{-4})] , \quad (5.15)$$

$$(0<<)\hat{z}_o < \hat{z}_{oB} \; (\equiv z_{oB}N_I/2\sqrt{2}\, G_B) .$$

We note again that this asymptotic series (5.15)* applies here only for "large" $\hat{z}_o$, smaller than some (large) value of $\hat{z}_{oB}$. For larger values of $\hat{z}_o$ ($>\hat{z}_{oB}$) we must use the second form of the PD, (5.7b), (5.8b), as explained earlier. This is done immediately below. Figures 5.1, 5.2, based on (5.11b), (5.15), shows typical PD's for $\hat{P}_1$, provided $|\hat{z}_o|<|\hat{z}_{oB}|$.

For P_{1-B-II}, cf. (5.7b), (5.8b), we use (5.9) directly, to obtain

$$P_1(z\geq z_o)_{B-II} \simeq \frac{1}{2}\{1-e^{-A_B}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\Theta[z_o/\sqrt{2}\,\hat{c}_{mB}]\} , \text{ (region II, cf. (5.5a))} \quad (5.16)$$

$$\begin{cases} \hat{c}_{mB}^2(=2\hat{\sigma}_{mB}^2) = (\frac{m}{\hat{A}_B}+\Gamma_B')/(1+\Gamma_B') , & (5.16a) \\ \Theta(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt , & (5.16b) \end{cases}$$

cf. (5.7c) and Eq. 3.16a), [Middleton, 1976]. Similarly, for (5.8b) we have at once

$$P_1(|z|\geq|z_o|)_{B-II} \simeq 1-e^{-A_B}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\Theta[|z_o|/\sqrt{2}\,\hat{c}_{mB}] , \; |z_o| > z_{oB} . \quad (5.17)$$

Note that (5.16), (5.17) have the proper behaviour as $z_o \to \pm\infty$, 0, viz:

$$\left.\begin{aligned} \lim_{z_o\to\pm\infty} P_1(z\geq z_o)_{B-II} &= 1/2-(1/2 \text{ or } -1/2) = 0, 1 \\ P_1(z\geq 0)_{B-II} &= 1/2 \end{aligned}\right\} \quad (5.18)$$

as was the case above for P_{1-B-I}, cf. (5.13).

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* See footnote on preceeding page.

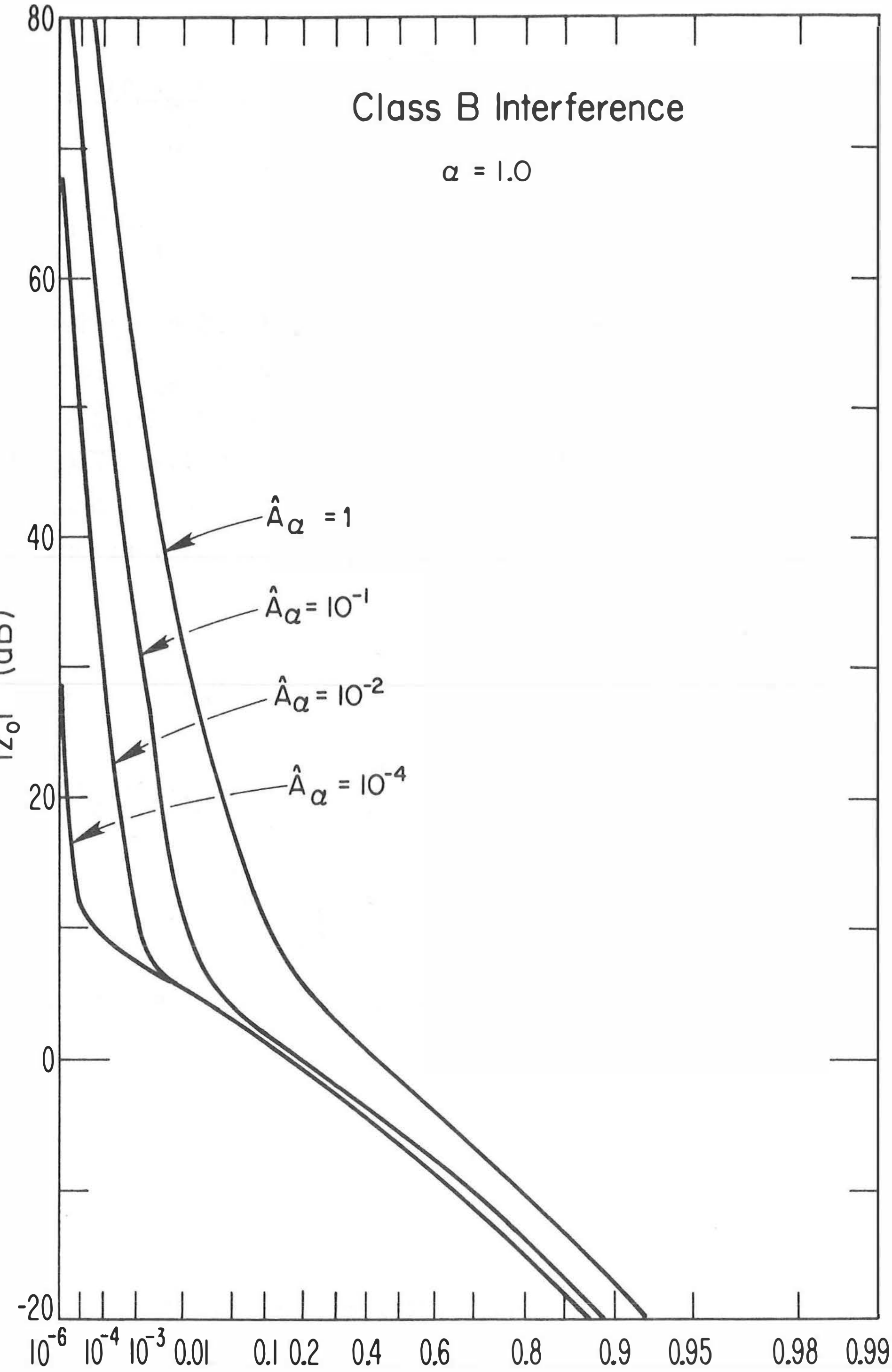

Figure 5.1 The amplitude distribution, $\text{Prob}(|\hat{z}|\geq|\hat{z}_o|)_I$, calculated for Class B interference for α = 1.0, for various values of the Index $\hat{A}_\alpha$, from Eqs. (5.11b), (5.15).

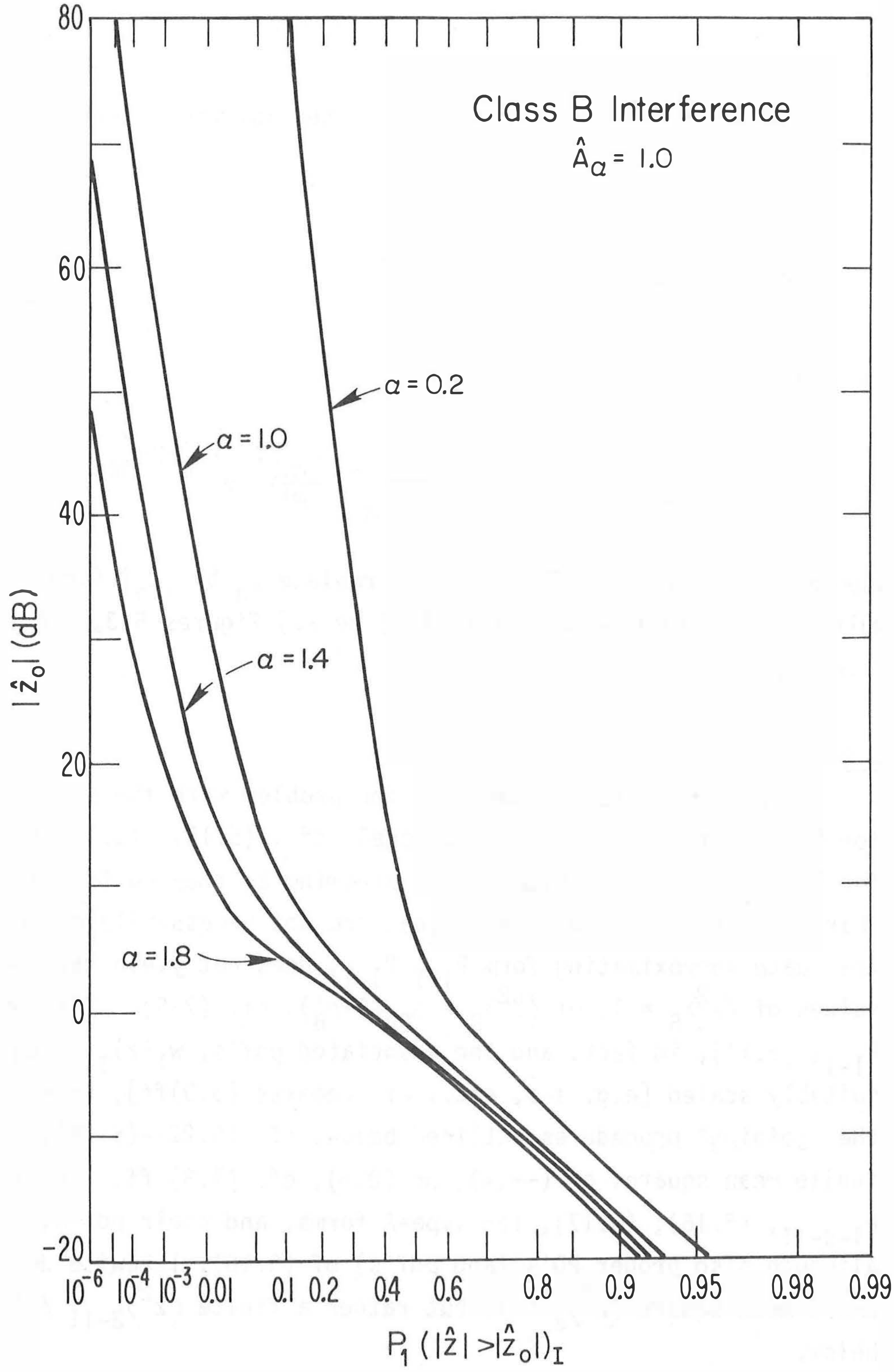

Figure 5.2 The amplitude distribution, $\mathrm{Prob}(|\hat{z}| \geq |\hat{z}_o|)_I$, calculated for Class B interference for $\hat{A}_\alpha = 1.0$, for various values of α, from Eqs. (5.11b), (5.15).

Finally, when z_o is large, we use the asymptotic relation [Middleton, 1960, Eq. (7.5)]

$$\Theta(y) \simeq 1 - \frac{e^{-y^2}}{\sqrt{\pi} y}\left[1 - \frac{1}{2y^2} + O(y^{-4})\right] , \qquad (5.19a)$$

to write (5.16) as

$$P_1(z \geq z_o)_{B-II} \simeq \frac{1}{\sqrt{2\pi}} \frac{e^{-A_B}}{z_o} \sum_{m=0}^{\infty} \frac{\hat{c}_{mB} A_B^m}{m!} e^{-z_o^2/2\hat{c}_{mB}^2}[1 - O(z_o^{-2})], \qquad (5.19b)$$

when $z_o^2 \gg 1$. [For (5.17) we simply replace z_o by $|z_o|$ (and z by $|z|$) and multiply the right member of (5.19b) by 2.] Figures 5.3, 5.4 illustrate P_{1-B-II}, Eq. (5.17).

5.2 The Composite Approximation:

As we have already remarked, the problem with the approximating results for P_{1-B} in the present Class B model, cf., (5.11), (5.15) for P_{1-B-I}, (5.16) for P_{1-B-II}, is that these forms, stemming as they do from approximate characteristic functions (4.23a,b), are not necessarily properly scaled in that each approximating form P_{1-I}, P_{1-II} does not yield the correct mean-square values of $\langle z^2 \rangle_B = 1$, or $\langle X^2 \rangle_B = \Omega_{2B}(1+\Gamma_B')$, cf. (7.5). The approximations P_{1-I}, (5.11), in fact, and the associated pdf's, $w_1(z)_{B-I}$, Eq. (6.7), although suitably scaled [e.g. $z \to \hat{z}$, etc., cf. remarks (5.9)ff], in accordance with the "joining" procedures outlined below, cf. (5.22)-(5.26), do not possess fnnite mean squares on $(-\infty,\infty)$, or $(0,\infty)$, cf. (7.3) ff. The approximations P_{1-B-II}, (5.16), (5.17), for Type-A forms, and their pdf's, w_{1-B-II}, Eq. (6.8), although also proper PD's (and pdf's) of (5.18), likewise do not yield the exact mean-square $\langle z^2 \rangle_B = 1$, but rather a finite $\langle z^2 \rangle_{B-II} \neq 1$, cf. (5.20a) below.

Accordingly, we must devise suitable methods of combining, or "joining", the B-I,II approximations, so as preserve the proper characteristics [(5.13),(5.18)] of the PD's, (and associated pdf's); provide the correct second moment, (7.7b); determine the various (six) B-model parameters, and thus effectvvely specify an acceptable approximation to P_{1-B}, and w_{1-B}, over

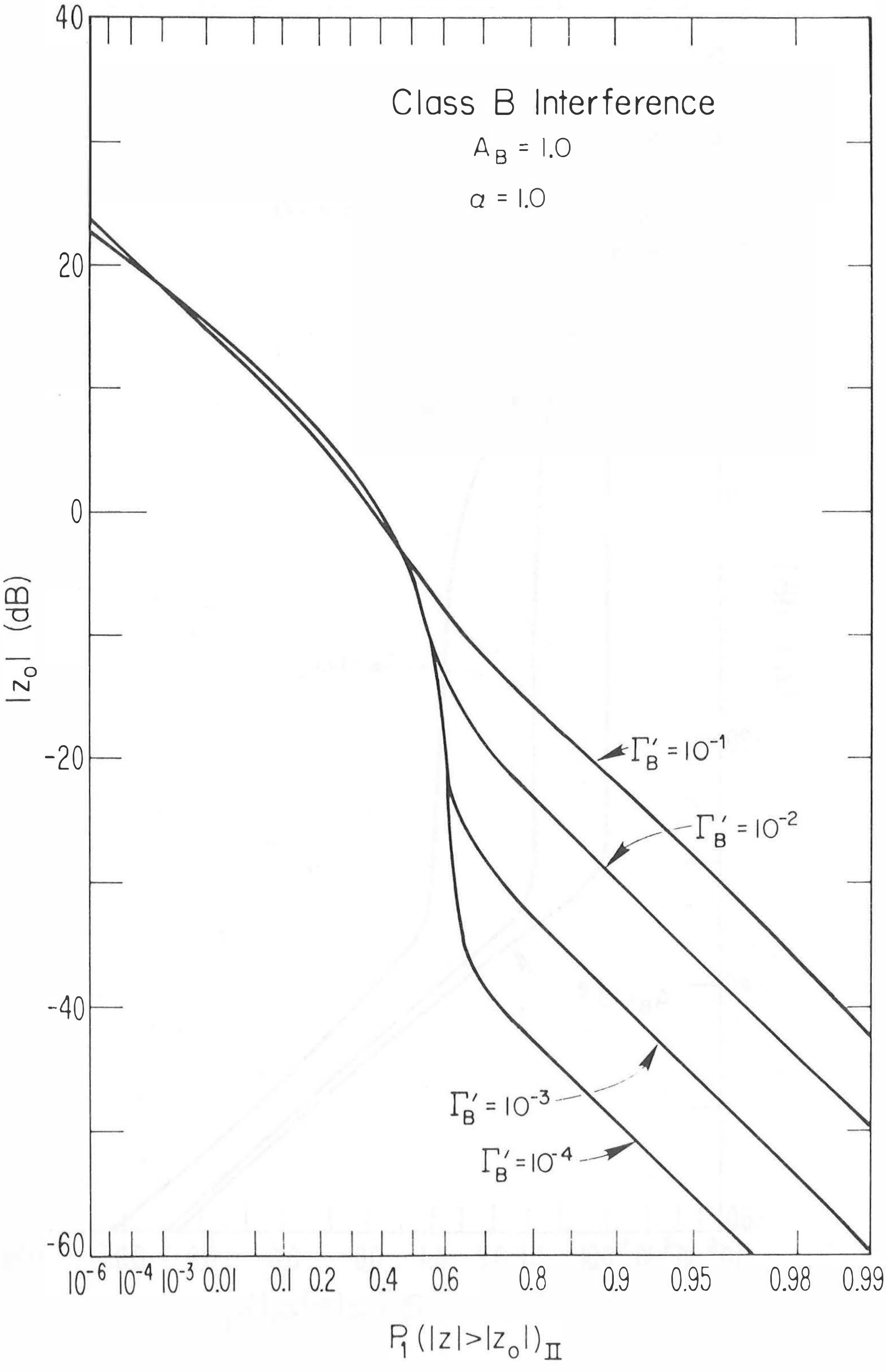

Figure 5.3 The amplitude distribution, $\text{Prob}(|z| \geq |z_o| (\geq z_{oB}))_{II}$, calculated for Class B interference for $A_B = 1.0$, $\alpha = 1.0$, for various values of Γ_B' from Eq. (5.17).

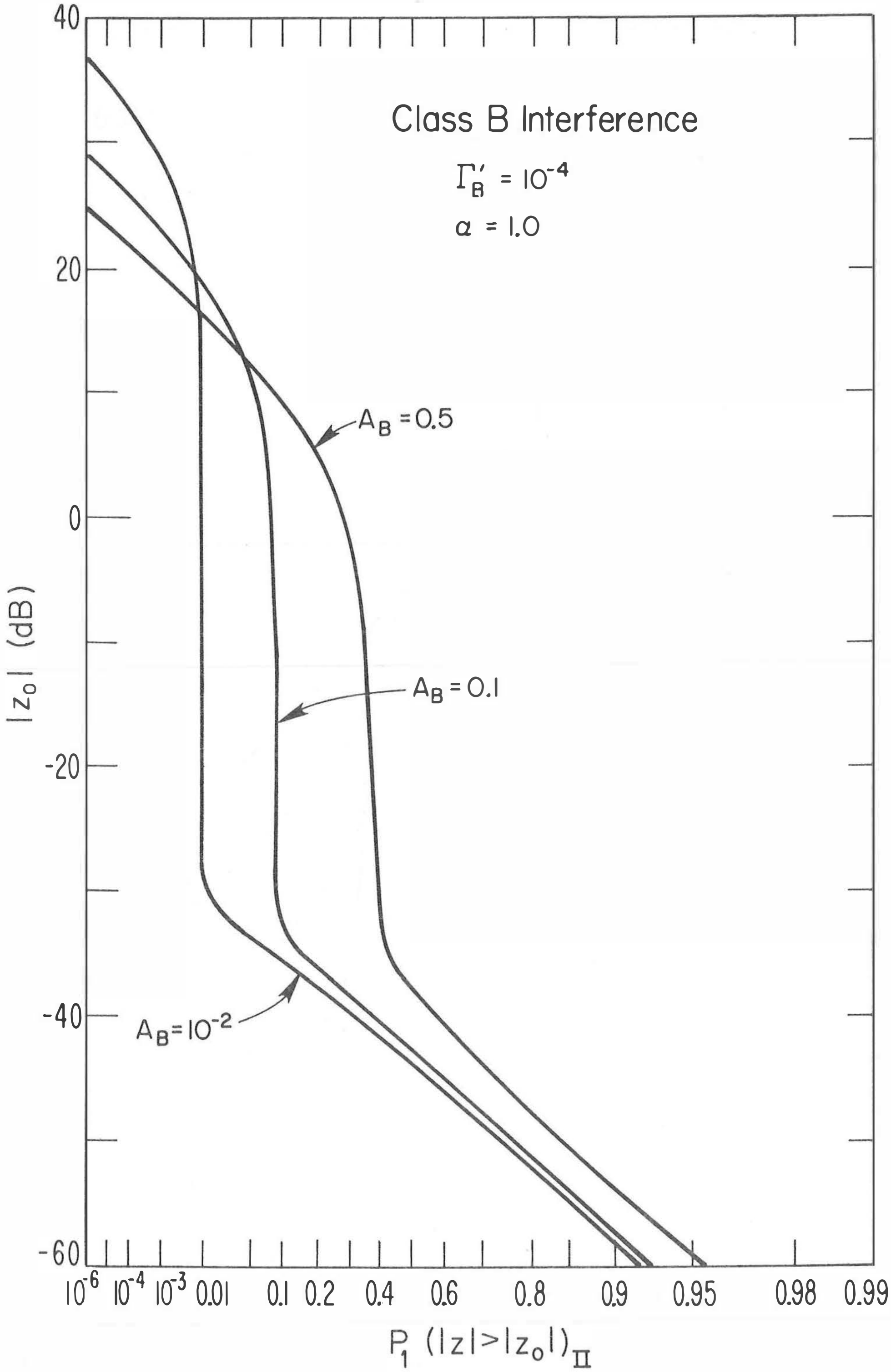

Figure 5.4 The amplitude distribution, $\text{Prob}(|z| \geq |z_o| \geq (z_{oB}))_{II}$, calculated for Class B interference for $\alpha = 1.0$, $\Gamma'_B = 10^{-4}$, for various values of A_B, from Eq. (5.17).

the entire range of the arguments (z,z_0). Two principal methods for accomplishing this are outlined below: Method No. 1 is a somewhat more simple, but approximate one, described earlier in Middleton [1976, Sec. 3.2-A; 1977, Sec. 3.2.1); Method No. 2 is precise, but involves more elaborate calculations. The former employs a suitably "normalized" pdf, $(w_{1-B-II})_{norm}$ for the B-II region, to determine $\langle z^2\rangle_B=1$, over the entire range $(-\infty,\infty)$, or $(0,\infty)$, of the random variables in question, and can thus yield errors of somewhat too small values of P_D for large (z,z_0), when Γ_B' is small. The latter method however, yields the correct values. The greater complexity of the latter lies in the explicit evaluation of $\langle z^2\rangle_B$ properly using the approximating pdf's for the appropriate regions B-I,II, cf. (5.23) below.

5.2.1 Method No. 1:

Let us outline Method No. 1 first, following the author's earlier analysis [Middleton, 1976, 1977]. Therefore, since the precise mean square is $\langle z^2\rangle_B = 1$, cf. (7.5), and since in this approach we now use w_{1-b-II} over all z to establish $\langle z^2\rangle_B = 1$, this means a "renormalization" of w_{1-B-II}, which is accomplished as follows:

(A). Calculate $\langle z^2\rangle_{II}$ on $(-\infty,\infty)$ according to (7.4), using w_{1-B-II}, Eq. (6.8), e.g.

$$\langle z^2\rangle_{II} = e^{-A_B}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\hat{c}_{mB}^2 = e^{-A_B}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}(m/\hat{A}_B+\Gamma_B')(1+\Gamma_B')^{-1} \tag{5.20}$$

$$= \frac{\left(\frac{4-\alpha}{2-\alpha}\right)+\Gamma_B'}{1+\Gamma_B'} = 4G_B^2\ (\neq 1),\ \text{cf. (5.10), with } \hat{A}_B = \left(\frac{2-\alpha}{4-\alpha}\right)A_B. \tag{5.20a}$$

The required normalization factor is thus $N_I^2 = (4G_B^2)^{-1}$, e.g.,

$$(w_{1-B-II})_{norm} = \frac{1}{4G_B^2}\ ,\quad w_{1-B-II} = N_{II}^2 w_1(z)_{B-II}. \tag{5.20b}$$

Using (6.8), applying N_{II}^2 directly to (5.16), (5.17) gives us now

$$P_1(z\geq z_0)_{B-II-norm} \cong \frac{1}{8G_B^2}\{1-e^{-A_B}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\,\Theta\,[z_0/\sqrt{2}\,\hat{c}_{mB}]\}, \tag{5.21a}$$

$$P_1(|z|\geq |z_0|)_{B-II-norm} \cong \frac{1}{4G_B^2}\{1-e^{-A_B}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\,\Theta\,[|z_0|/\sqrt{2}\,\hat{c}_{mB}]\}, \tag{5.21b}$$

where the domains of applicability (for z_0 vs z_{0B}) are as indicated in (5.5a,b) above. [These amplitude relations, (5.21a,b), are the analogues of (3.17), [Middleton, 1976], for the envelope.]

(B). The case of w_{1-B-I}, Eq. (6.7), requires a different approach, since $\langle z^2\rangle_I$ on $(-\infty,\infty)$, or $(0,\infty)$ is infinite $(0<\alpha<2)$, cf. Sec. 7.3. Instead of normalizing with respect to the second moment, [as is done in (i) above, which is, of course, not possible here], we must scale z_0 (and z) according to (5.11d), e.g., $z_0 \to \hat{z}_0' = N_I z_0'$, with $z_0' = X_0/2\sqrt{2}\,G_B\sqrt{\Omega_{2B}(1+\Gamma_B')}$, so that $\hat{z}_0 = z_0 N_I/2\sqrt{2}\,G_B$. The rationale for this is

1). The observation that the approximating P_{1-B-I} (and w_{1-B-I}) must have the same values in the gauss region ($z_0^2<<1$), where $P_1>(0.9)$, or more, as does the precise PD, P_{1-B} (and pdf w_{1-B}), derived from the exact (but intractable) c.f. [(2.89), $r\to\xi$, in Middleton [1976]]. Hence we have (5.11) above.

∴ 2). The scaling factor N_I is to be determined by fitting the two approximate PD's, P_{1-I}, P_{1-II}, together by the procedure outlined below, which is based on the canonical properties of the Class B model generally.

The "joining procedure" for $P_{1-I,II}$ (and therefore for $w_{1-I,II}$) now directly parallels that stated in [Middleton, Sec. 3.2B, 1976; Sec. 3.2.1, 1977], and uses (5.11), (5.20) specifically: we set

(i). $P_{1-I} = P_{1-II}$ in the gauss region (e.g. $|z_0|\sim 0$ or small). (5.22a)
Equality of the two approximations in this region is required, since both represent the same (small-) amplitude behaviour, characteristic of all PD's here, and, of course, since both are approximations to the same, single, exact PD;

(ii). $\frac{dP_{1-I}}{dz_o} = \frac{dP_{1-II}}{dz_o}$ in the gauss region: for the same reasons, both approximating PD's must have the same (constant) slopes, $(P_1 > 0.9)$; (5.22b)

(iii). $P_{1-I} = P_{1-II}$ at $\pm z_{oB}$, i.e. at the "bendover" point (cf. Fig. 5.5), $z_{oB}(>0)$, where the two approximations are joined together. This condition is required for (5.5) to hold, as shown in Appendix A, cf. Eq. (A.1-6), et seq. It also insures that the PD here remains continuous. The point z_{oB} is empirically determined from the data, i.e. from the experimental APD, or exceedance probability curve $P_1(z \geq z_o)_{exp}$. (5.22c)

(iv). $\left(\frac{dP_{1-I}}{dz_o} = \frac{dP_{1-II}}{dz_o}\right)_{\pm z_{oB}}$: the (finite) slopes of the PD's must also be equal at the bend-over point(s) $\pm z_{oB}$). This insures that the associated pdf's are continuous at $(\pm z_{oB})$, and that the PD's are smooth here. (5.22d)

(v). $\left(\frac{d^2P_{1-I}}{dz_o^2} = \frac{d^2P_{1-II}}{dz_o^2}\right)_{\pm z_{oB}}$ $(\neq 0)$: (5.22e)

this is required by the condition that the pdf's be not only continuous but "smooth", i.e. have a common tangent, at $\pm z_{oB}$. This, in turn, is required by the fact that the exact pdf is not only continuous but smooth, as well.

(vi). $z_{oB}(>0)$: this is the point of inflection $(d^2P_1/dz_{oB}^2 = 0)$ of the exact P_1, and is determined as such (usually by inspection of the P_1-data), with $P_1 \to P_{1-expt}$, cf. Fig. 5.5. This value of z_o $(= \pm z_{oB})$ cannot be determined analytically from either approximating form, $P_{1-I,II}$. (5.22f)

The five relations, (i)-(v), with the sixth, $\langle z^2 \rangle_{II} = 1$, e.g., $\langle X^2 \rangle_{II} = 2\Omega_{2B}(1+\Gamma_B') = \langle X^2 \rangle_{expt}$ cf. (i), Eqs. (5.20)-(5.21b), are sufficient to

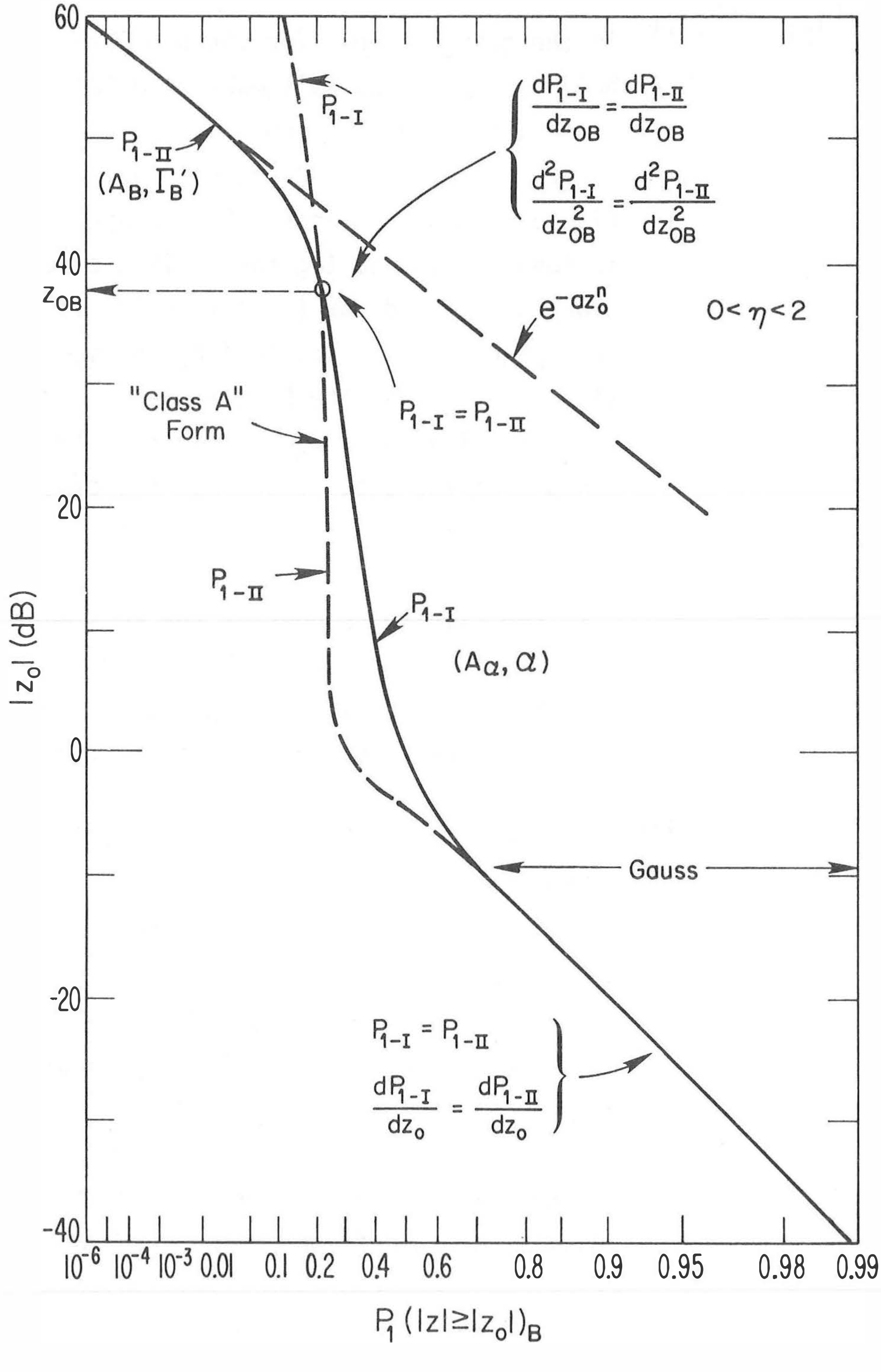

Figure 5.5 Schema of P_{1-B}, Eq. (5.27b), obtained by joining the two approximating forms (5.11b), (5.17), according to the procedures (5.22), Method 1, or (5.26), Method 2.

determine the six global parameters $(A_B, \Omega_{2B}, \Gamma'_B, \alpha, A_\alpha, N_I)$, as long as z_{oB} is empirically available.

5.2-2: Method 2:

As noted above, this is a precise version of (i)-(v) above, Eq. (5.22), with the sixth, (approximate condition in Method 1) replaced by the "exact" relation $\langle z^2\rangle_B = \langle z^2\rangle_{B\text{-region I}} + \langle z^2\rangle_{B\text{-region II}} - 1$, e.g., (7.7b) in (7.2a). Now we use the proper, approximating PD's (and pdf's), (5.11), (5.16), (and (6.7), (6.8)) in (i)-(vi) above, cf. (5.22). These relations here are then explicitly:

(A). $\langle z^2\rangle_B = 1 = \langle z^2\rangle_{B\text{-region I}} + \langle z^2\rangle_{B\text{-region II}}$

$$= \int_{-z_{oB}}^{z_{oB}} z^2 w_1(z)_{B-I} dz + \left(\int_{-\infty}^{-z_{oB}} + \int_{z_{oB}}^{\infty}\right) z^2 w_1(z)_{B-II} dz \tag{5.23a}$$

$$= 2\int_0^{z_{oB}} z^2 w_1(z)_{B-I} dz + 2\int_{z_{oB}}^{\infty} z^2 w_1(z)_{B-II} dz \,, \tag{5.23b}$$

this last because of the symmetry of the pdf's, which also, cf. (6.8a)', allows us to express (5.23b) as

$$\langle z^2\rangle_B = \int_0^{z_{oB}} z^2 w_1(|z|)_{B-I} d|z| + \int_{z_{oB}}^{\infty} z^2 w_1(|z|)_{B-II} d|z| \,, \tag{5.23c}$$

in terms of the pdf for $|z|$. Then, since $w_1(z)_{B-I}dz = w_1(\hat{z})d\hat{z}$, where $w_1(\hat{z})$ is the "scaled" pdf for region B-I, cf. (6.7a), and $\hat{z} = z_0 N_I / 2\sqrt{2}\, G_B$, cf. (5.11d), we get, using $\hat{w}_1(\hat{z})$, (6.7a),

$$\langle z^2\rangle_{B\text{-region I}} = 2\int_0^{z_{oB}} z^2 w_1(z)_{B-I} dz = 2\int_0^{\hat{z}_{oB}} \left(\frac{8G_B^2}{N_I^2}\right) \hat{z}^2 \hat{w}_1(\hat{z}) d\hat{z} \,. \tag{5.23d}$$

The desired condition using the correct mean square then reduces to

$$\boxed{1 = \frac{8G_B^2}{N_I^2}\int_0^{\hat{z}_{oB}(>0)} \hat{z}^2\hat{w}_1(|\hat{z}|)d|\hat{z}|+\int_{z_{oB}}^{\infty} z^2w_1(|z|)_{B-II}d|z|.} \tag{5.23e}$$

We can evaluate the second integral partially in closed form, using (6.8), (6.8a)', to get directly

$$\left\langle z^2\right\rangle_{B\text{-region II}} = \int_{z_{oB}}^{\infty} z^2w_1(|z|)_{B-II}d|z|$$

$$= e^{-A_B}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\left\{\sqrt{\frac{2}{\pi}}\, z_{oB}\hat{c}_{mB}e^{-z_{oB}^2/2\hat{c}_{mB}^2}\right.$$

$$\left.+\hat{c}_{mB}^2\left(1-\Theta[z_{oB}/\sqrt{2}\,\hat{c}_{mB}]\right)\right\}, \tag{5.24a}$$

$$= 4G_B^2 + e^{-A_B}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\left\{\sqrt{\frac{2}{\pi}}\, z_{oB}\hat{c}_{mB}e^{-z_{oB}^2/2\hat{c}_{mB}^2}\right.$$

$$\left.-\hat{c}_{mB}^2\;\Theta\,[z_{oB}/\sqrt{2}\,\hat{c}_{mB}]\right\}. \tag{5.24b}$$

[We remark that these results necessarily reduce to that of (i), Method 1, above, when we set $z_{oB}\to 0$ in the calculation of $\left\langle z^2\right\rangle_{II}$, cf. (5.20)-(5.21).] The first integral in (5.23e) can be obtained formally on a term-wise basis. We get

$$\left\langle z^2\right\rangle_{B\text{-region I}} = \frac{8G_B^2}{N_I^2}\cdot\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\hat{A}_\alpha^n}{n!}\,\Gamma\left(\frac{n\alpha+1}{2}\right)\int_0^{\hat{z}_{oB}} x^2\,{}_1F_1\left(\frac{n\alpha+1}{2};1/2;-x^2\right)dx\;, \tag{5.25}$$

where

$$\int_0^{\hat{z}_{oB}} x^2\,{}_1F_1\left(\frac{n\alpha+1}{2}\;;1/2;-x^2\right)dx = \sum_{\ell=0}^{\infty}\frac{\left(\frac{n\alpha+1}{2}\right)_\ell\hat{z}_{oB}^{2\ell+3}}{\ell!(1/2)_\ell(2\ell+3)}\;. \tag{5.25b}$$

Accordingly, (5.24) and (5.25) in (5.23e) yield one relationship among the six global parameters of this Class B model.

The other five relations are obtained as above (5.22a-f). We have explicitly here, for Method 2:

(i). $[P_{1-I}=P_{1-II}]_{|z_o|<<1}$ (cf. 5.22a) gauss region $: \frac{1}{2} - \frac{\hat{z}_o}{\pi}\sum_{n=0}^{\infty}(-1)^n\Gamma(\frac{n\alpha+1}{2})\hat{A}_\alpha^n/n!$

$$= \frac{1}{2}\left\{1-e^{-A_B}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\frac{z_o}{\hat{c}_{mB}}\sqrt{\frac{2}{\pi}}\right\},$$

which becomes

$$\frac{N_I}{2\sqrt{\pi}\,G_B}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\frac{n\alpha+1}{2})\hat{A}_\alpha^n}{n!} = e^{-A_B}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\frac{1}{\hat{c}_{mB}}, \tag{5.26a}$$

for both z_o and $|z_o|$, cf. (5.11b), (5.17).

(ii). $\left[\frac{dP_{1-I}}{dz_o} = \frac{dP_{1-II}}{dz_o}\right]_{|z_o|<<1}$ cf. (5.22b), gauss region, also for both z_o and $|z_o|$ $: \frac{N_I}{G_B^2\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Gamma(\frac{n\alpha+3}{2})\hat{A}_\alpha^n = e^{-A_B}\sum_{n=0}^{\infty}\frac{A_B^m}{m!\hat{c}_{mB}^2}$

(iii). $[P_{1-I}=P_{1-II}]_{\pm z_{oB}}$ cf. (5.22c) $: \frac{\hat{A}_\alpha\Gamma(\frac{\alpha+1}{2})\hat{z}_{oB}^{-\alpha}}{\sqrt{2}\,\Gamma(1-\alpha/2)}[1+0(z_{oB}^{-2},\alpha)]$

$$\simeq \frac{e^{-A_B}}{z_{oB}}\sum_{m=0}^{\infty}\frac{\hat{c}_{mB}A_B^m}{m!}e^{-z_{oB}^2/2\hat{c}_{mB}^2}[1-(z_{oB}^2)],$$

where (5.15) and (5.19b) are used.

(iv). $\left[\frac{dP_{1-I}}{dz_o} = \frac{dP_{1-II}}{dz_o}\right]_{\pm z_{oB}}$ cf.(5.22d) $: \frac{\hat{A}_\alpha\alpha\Gamma(\frac{\alpha+1}{2})}{\sqrt{2}\Gamma(1-\alpha/2)}z_o^{-\alpha-1}[1+0(\hat{z}_{oB}^{-2},\alpha)]$

$$\simeq \frac{1}{z_{oB}^2}e^{-A_B}\sum_{m=0}^{\infty}(\hat{c}_{mB}+\frac{1}{2\hat{c}_{mB}})\frac{A_B^m}{m!}e^{-z_{oB}^2/2\hat{c}_{mB}^2}$$

$$[1-0(z_{oB}^{-2})], \tag{5.26d}$$

again where (5.15) and (5.19b) are used

(v). $\left[\frac{d^2P_{1-I}}{dz_o^2} = \frac{d^2P_{1-II}}{dz_o^2}\right]_{\pm z_{oB}}$: cf. (5.22e), which insures that the pdf's at $\pm z_{oB}$ are "smooth", etc. [The explicit analytic form is obtained from (5.15) and (5.19b) by differentiation as before.]

(vi). $\pm z_{oB}$: the "bend-over" points again are empirically determined and used in the above, cf. (5.22f).

We remark, also, that when the relations (iii)-(v) are used, we may need the next set of "correction terms" in these various asymptotic developments. Since Class B models involve six (ultimately) independent parameters, fixing any three in the above enables us to determine the other three, from any three of the (i)-(v), and (5.23) etc. Finally, we note that the "Class A" form (II) is coupled to the "Class B" form (I) through the Class B parameter, α, and vice versa through the "Class A" parameter, Γ_B', appearing in G_B, common to both approximations I,II. In any case, we have, for either Method:

$$\begin{cases} P_{1-B} = P_{1-I}, \quad -z_{oB}<z_o<z_{oB} \; ; \quad P_{1-B} = P_{1-II} \; ; \\ \qquad\qquad -\infty<z<-z_{oB}, \; z_{oB}<z<\infty. \\ \quad = P_{1-I}, 0 < |z| < z_{oB} \; ; \quad = P_{1-II}, \quad |z_o|<|z| < \infty. \end{cases} \qquad \begin{matrix} (5.27 \\ (5.27b) \end{matrix}$$

Typical distributions are shown in Figs. 5.6, 5.7.

5.3 Hall-Type Models:

As in the envelope cases treated earlier [cf., Sec. 3.2B, [Middleton, 1976]], we can also obtain a Hall model form [Hall, 1966] from the P_{1-I} approximation used in the gauss and intermediate region $0 \leq |z_o| \leq |z_{oB}|$. This is achieved provided (1), we omit the gaussian contribution in the c.f. (5.7a), or (5.8a), e.g., set $\Delta\sigma_G^2 \to 0$ therein, and (2) set $z_{oB}\to\infty$. The resulting expression for the PD is accordingly [cf. (5.7a)]

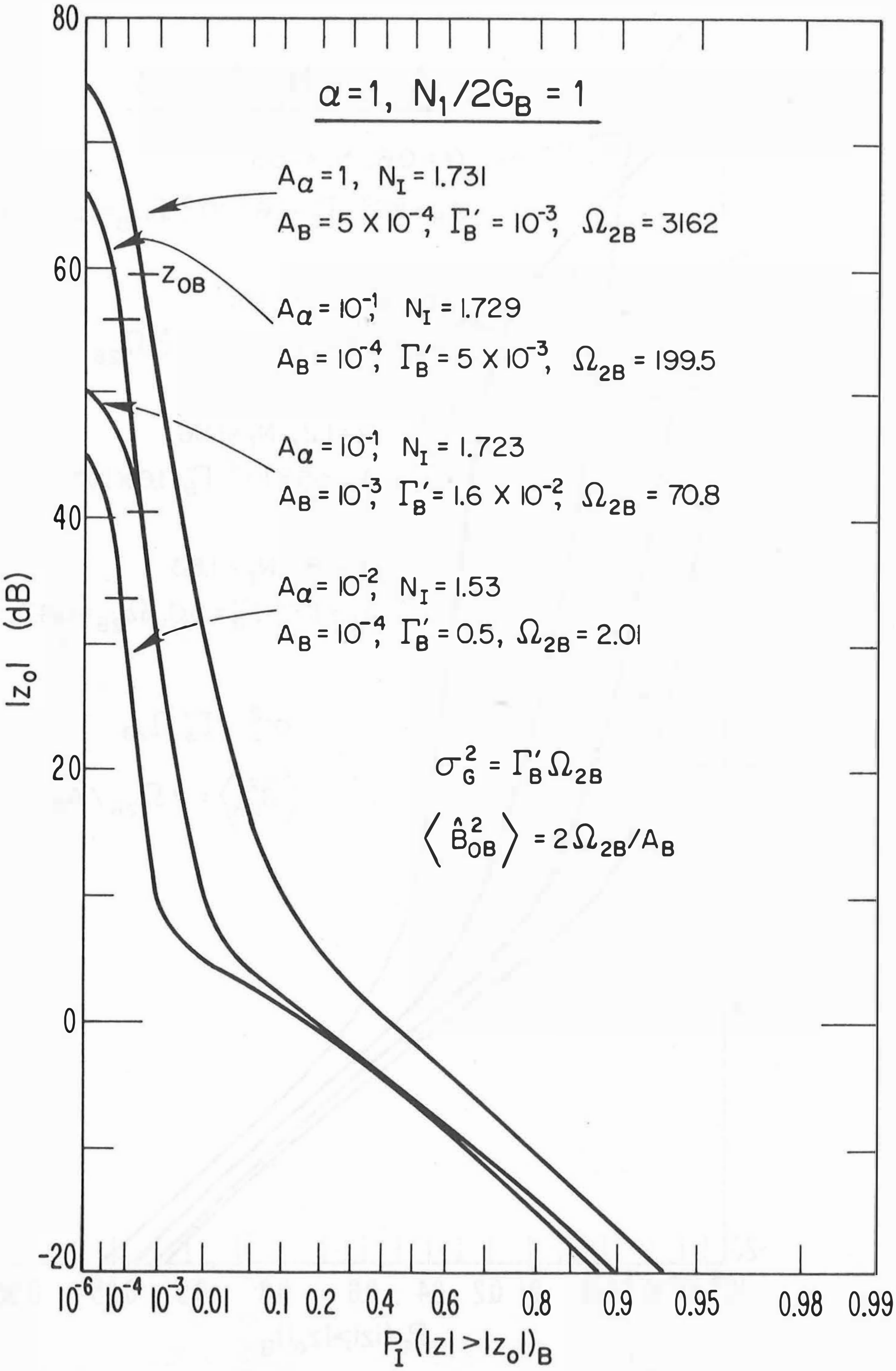

Figure 5.6 The (complete) amplitude distribution $P_1(|z| \geq |z_0|)_B$, Eq. (5.27b), calculated for Class B interference for various A_α, given α, from Eqs. (5.11b), (5.17), according to the procedures of (5.26).

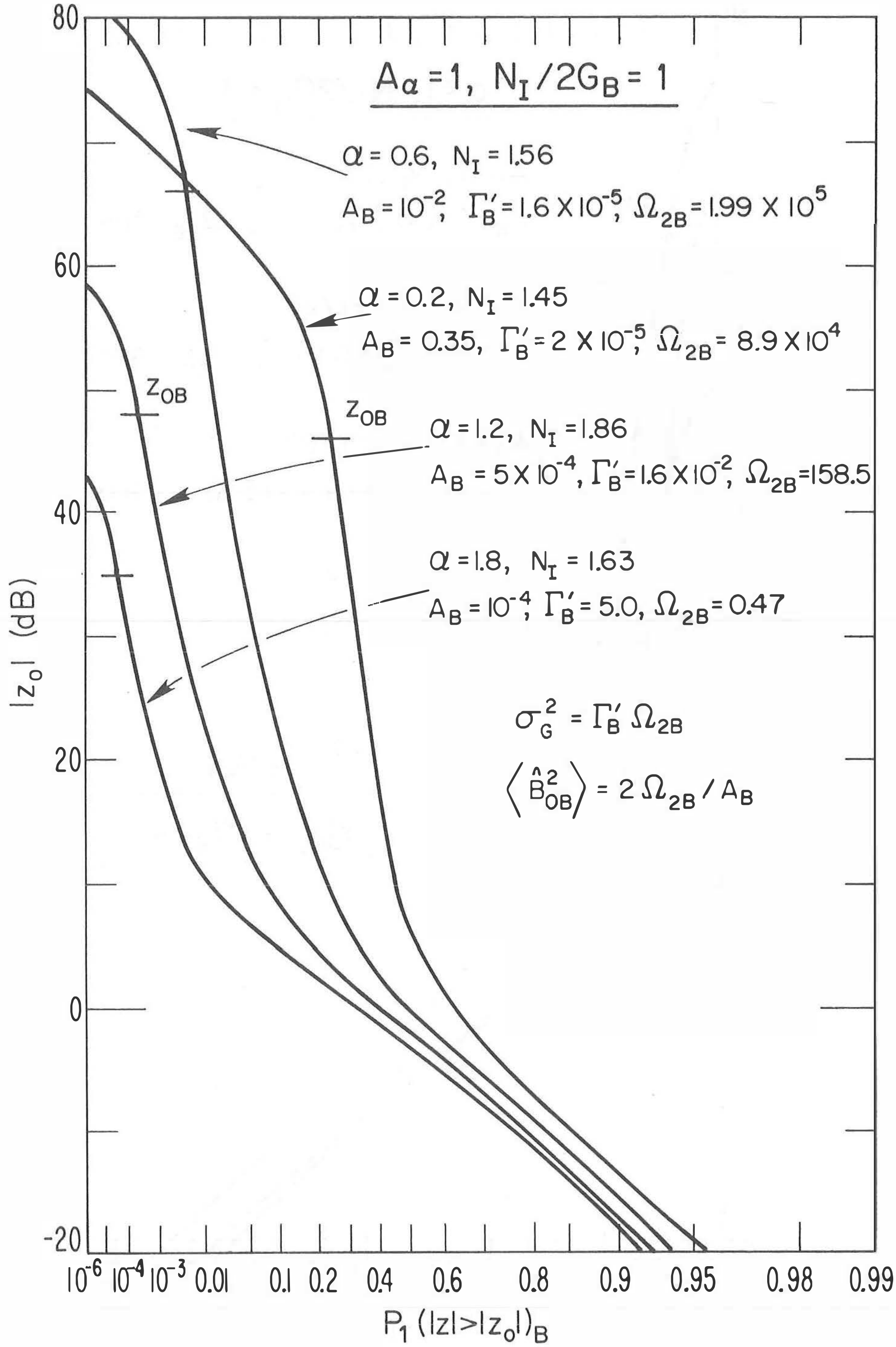

Figure 5.7 The (complete) amplitude distribution $P_1(|z| \geq |z_o|)_B$, Eq. (5.27b), calculated for Class B interference for various α, given A_α, from Eqs. (5.11b), (5.17) according to the procedures of (5.26).

$$P_1(z\geq z_o)_{B-I|Hall} \simeq \frac{1}{2} - \sqrt{\frac{z_o}{2\pi}} \int_0^\infty \lambda^{-1/2} J_{1/2}(z_o\lambda) e^{-b_{1\alpha}A_B(\hat{a}\lambda)^\alpha} d\lambda \quad . \tag{5.28}$$

We may proceed to evaluate (5.28), following the approach used on pp. 100-103 [Middleton, 1976]. Let us start with the form convenient for small values of $|z_o|$ ($\leq z_{oB}$) and use the following transformations:

$$\hat{B}_\alpha \equiv b_{1\alpha}A_B\hat{a}^\alpha \; (=A_\alpha 2^{-\alpha/2}, \text{ cf. } (5.11c)) \; ; \quad z_\alpha = \hat{B}_\alpha\lambda^\alpha \tag{5.29a}$$

$$\therefore \quad \lambda = (z_\alpha/\hat{B}_\alpha)^{1/\alpha} \; ; \quad d\lambda = z_\alpha^{1-\alpha/\alpha} dz_\alpha/\alpha\hat{B}^{1/\alpha} \tag{5.29b}$$

and

$$z_o^* \equiv z_o/\hat{B}_\alpha^{1/\alpha}. \tag{5.29c}$$

We obtain for (5.28)

$$P_1(z\geq z_o)_{B-I} \simeq \frac{1}{2} - \frac{1}{\alpha}\sqrt{\frac{z_o^*}{2\pi}} \int_0^\infty z_\alpha^{1/2\alpha} J_{1/2}(z_o^* z_\alpha^{1/\alpha})\, e^{-z_\alpha} dz_\alpha \tag{5.30}$$

Next, we use the Barnes-integral representation for $J_{1/2}$:

$$J_{1/2}(z_o^* z_\alpha^{1/\alpha}) = \int_\Gamma \frac{\Gamma(-s)}{\Gamma(s+3/2)} \left(\frac{z_o^*}{2}\right)^{2s+1/2} z_\alpha^{(2s+1/2)/\alpha} \frac{ds}{2\pi i} \quad , \tag{5.31}$$

cf. Eq. (13.106), [Middleton, 1960], when Γ is the contour $(-\infty+c, i\infty+c)$, with $c(<0)$ so chosen that the integral over z_α in (5.30) is convergent at $z_\alpha=0$, e.g.

$$\int_0^\infty z_\alpha^{\frac{1}{2\alpha}+\frac{2s+1/2}{\alpha}-1} e^{-z_\alpha} dz_\alpha = \Gamma\left(\frac{2s+1}{\alpha}\right), \quad \text{Re } s > -1/2 \; ; \quad -1/2 < c < 0. \tag{5.32}$$

Applying (5.32) to (5.30), (5.31) then gives

$$P_1(z>z_0)_{B-I} \simeq \frac{1}{2} - \frac{1}{\alpha}\sqrt{\frac{z_0^*}{2\pi}}\int_\Gamma \frac{\Gamma(-s)\Gamma(\frac{2s+1}{\alpha})(z_0^*)^{2s+1/2}}{\Gamma(s+3/2)2^{2s+1/2}}\frac{ds}{2\pi i}, \tag{5.32a}$$

which is evaluated (at the (simple) poles s=0,1,2,...) to yield directly

$$\boxed{\begin{aligned} \therefore\ P_1(z\geq z_0)_{B-I|Hall} &\simeq \frac{1}{2} - \frac{1}{\pi\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\frac{2n+1}{\alpha})(z_0^*)^{2n+1}}{n!(3/2)_n 2^{2n}}, && (5.33b)\\ &\simeq \frac{1}{2} - \frac{\sqrt{2}}{\pi\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\frac{2n+1}{\alpha})}{n!(3/2)_n 2^n}\frac{z_0^{2n+1}}{A_\alpha^{\frac{2n+1}{\alpha}}}. && (5.33c)\end{aligned}}$$

[Note that in the gauss region ($|z_0| \ll 1$), Eqs. (5.33a,b) exhibit the characteristic gauss form (in the PD) $\sim z_0$, cf. (5.11), as $z_0, \hat{z}_0 \to 0$.]

For large values of $z_0 (<z_{0B} \to \infty)$ (or small values of A_α), we return to (5.28) and use a Barnes integral representation for $\exp(-\hat{B}_\alpha \lambda^\alpha)$, viz:

$$e^{-\hat{B}_\alpha\lambda^\alpha} = \int_\Gamma \Gamma(-s)\hat{B}_\alpha^s\lambda^{\alpha s}\frac{ds}{2\pi i}, \tag{5.34}$$

to reëxpress (5.28) as

$$P_1(z\geq z_0)_{B-I|Hall} \simeq \frac{1}{2} \mp \frac{1}{\sqrt{2\pi}}\int_\Gamma \Gamma(-s)|z_0^*|^{-\alpha s}\frac{ds}{2\pi i}\int_0^\infty y^{\alpha s-1/2}J_{1/2}(y)dy, \tag{5.35}$$

$$-1 < \text{Re}\ \alpha s < 0,$$

where we have employed the transformation $z_0\lambda = y$ and $(I) \to z_0 \to {}^{+\infty}_{-\infty}$ here. Again, we use [Watson, (1944), p. 391] to evaluate the y-integral:

$$\int_0^\infty J_\nu(t)t^{\mu-\nu-1}dt = \Gamma(\mu/2)/\Gamma(\nu\cdot\mu/2+1)2^{\nu-\mu+1}, \quad \text{Re}\ \mu < \text{Re}(\nu+3/2) \tag{5.36}$$

with $\nu = 1/2$, $\mu = \alpha s+1$ here, and $\therefore$ $0 < \text{Re}(\alpha s+1) < 1 (<2)$, as required. Equation (5.35) now becomes

$$P_1(z\geq z_o)_{B-I|Hall} \simeq \frac{1}{2} \mp \frac{1}{\sqrt{\pi}} \int_\Gamma \frac{\Gamma(-s)\Gamma(\frac{\alpha s+1}{2})|z_o^*|^{-\alpha s}}{\Gamma(1-\alpha s/2)a^{1-\alpha s}} \frac{ds}{2\pi i}$$

$$\simeq \frac{1}{2} \mp \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{n\alpha+1}{2})}{n!\ \Gamma(1-\alpha n/2)} \frac{2^{\alpha n/2} A_\alpha^n}{|z_o|^{\alpha n}} ,$$

$$\boxed{\therefore\ P_1(z\geq z_o)_{B-I|Hall} \simeq (\frac{1}{2} \mp \frac{1}{2}) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\Gamma(\frac{n\alpha+1}{2}) 2^{\alpha n/2} A_\alpha^n}{n! \quad \Gamma(1-\alpha n/2)|z_o|^{\alpha n}}} \tag{5.37}$$

which shows, as expected, that

$$\lim_{z_o\to\infty} P_1 \to 0 \ ; \lim_{z_o\to-\infty} P_1 \to 1 \ , \quad (|z_{oB}|\to\infty,\ \ |z_{oB}|>z_o \text{ here}), \tag{5.37a}$$

formally. Thus, we can use the relations (5.37) for large values of $|z_o|$. Equations (5.33b,c), (5.37) constitute the Hall-type PD's for Class B interference. [As before, [Middleton, 1976], there are no Hall-type models for Class A noise. [These results are generalizations of the Hall approach, based now on detailed physical considerations, i.e., those leading to the results of Sec. 4 above for the c.f.'s.]

Finally, in the special case $\alpha=1$ (as in the envelope case, cf. pp. 102-103, [Middleton, 1976]) we can sum the series (5.33c), or (5.37). Thus, we have (from (5.33c)), all z_o,

$$\underline{\alpha=1}: \quad P_1(z\geq z_o)_{B-I|Hall} \simeq \frac{1}{2} - \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(2n+1) z_o^{2n+1}}{n!(3/2)_n 2^n\ A_{\alpha=1}^{2n+1}} \tag{5.38}$$

$$= \frac{1}{2} - \frac{\sqrt{2}}{\pi} \frac{z_o}{A_{\alpha=1}} \sum_{n=0}^{\infty} \frac{(-1)^n (1/2)_n}{(3/2)_n} (\frac{\sqrt{2}\ z_o}{A_{\alpha=1}})^{2n} \frac{(1)_n}{n!} \tag{5.38a}$$

$$\therefore\ P_1(z\geq z_0)\Big|_{\alpha=1} = \frac{1}{2} - \frac{1}{\pi}\left(\frac{\sqrt{2}\,z_0}{A_\alpha}\right){}_2F_1\left(1/2,1;3/2;-\frac{2z_0^2}{A^2_{\alpha=1}}\right) \\ = \frac{1}{2}\left\{1-\frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\,z_0}{A_\alpha}\right)\right\} . \qquad (5.38b)$$

Here the corrdsponding pdf is at once

$$w_1(z)_{I\text{-Hall}} = -\frac{dP_1}{dz_0}\Big|_{z_0\to z} = \frac{1}{\pi}\left(\frac{\sqrt{2}\,A^{-1}_{\alpha=1}}{1+2z^2/A^2_{\alpha-1}}\right) ,\quad \alpha=1 . \qquad (5.39)$$

It is at once evident that only the $\langle|z|^\beta\rangle$-moment, $-1<\beta<1$, exists for this Hall model, and consequently $\langle z^2\rangle\to\infty$, as expected from our general analysis [cf. Sec. 7]. The PD, (5.38b), and pdf, (5.39), correspond to the Hall case $\theta_{Hall}= 2$, for amplitudes [cf. Spaulding, Middleton, 1976, Eq. 2.33 et seq.]

Note that P_1, $(0<\alpha<2)$, is monotonically decreasing as $z_0\to+$, and that $\lim_{z_0\to\pm\infty} P_1=(0,1)$, with $P_1(z\geq 0) = 1/2$, as required of a proper PD. It is, however, an inappropriate approximate form when $\Delta\sigma_G$ is at all comparable to $(b_{1\alpha}A_B)^{1/\alpha}$. Also, it is not applicable for very large z_0, since the true, (i.e., physical) pdf's must fall off fast enough to guarantee the existence of a finite second moment [cf. comments in Sec. (2.7A), Middleton, 1976]. In any case, Hall-type PD's and pdf's are not possible for Class A interference, as noted earlier. Moreover, because of the improper scaling of these PD's, e.g., P_{1-B-I}, generally, we must replace z, z_0, by $N_I'z$, $N_I'z_0$, where the scaling factor N_I' along with the four other parameters $(A_\alpha,\alpha,\Gamma_B',\Omega_{2B})$, are determined by the procedure outlined in Section 2 (and in Section 6C, [Middleton, 1976]. [In the case of the Hall model here where $\alpha=1$, cf. (5.38), (5.39), there are only the four parameters $(N_I',A_\alpha,\Gamma_B',\Omega_{2B})$ to be established.]

6. Probability Densities: $w_1(z)_B$:

The associated probability densities (pdf's) for the approximations $w_1(z)_{I,II}$ for these Class B interference models are now readily obtained, either from

$$w_1(z) = -\left.\frac{dP}{dz_o}\right|_{z_o \to z} \tag{6.1}$$

or by using (the normalized) form of (4.6), viz:

$$w_1(z)_B = \frac{1}{\pi}\int_{0-}^{\infty} \cos \lambda z \; F_1(i\hat{a}\lambda)_X d\lambda \tag{6.2}$$

for these symmetric pdf's [derived from (4.6) with the help of (5.3) and the relation $w_1(z)_B = w_1(X(z))|dX/dz|$]. Here we have, in the approximate form:

$$\boxed{\left.\begin{aligned} w_1(z)_B &\cong w_1(z)_{B-I} \; ; \quad -z_{oB} < z < z_{oB} \, , \\ &\cong w_1(z)_{B-II} \, , \quad |z| > z_{oB} \, , \end{aligned}\right\}} \tag{6.3}$$

corresponding to the associated PD's, (5.27). For example, (6.2) becomes

$$w_1(z)_{B-I,II} \cong \frac{1}{\pi}\int_{0-}^{\infty} \cos \lambda z \; F_1(i\hat{a}\lambda)_{b-I,II} d\lambda \tag{6.4a}$$

$$\cong \frac{1}{\pi}\int_{0-}^{\infty} \cos \lambda z \begin{Bmatrix} e^{-b_{1\alpha}A_B\hat{a}^{\alpha}|\lambda|^{\alpha}-\Delta\sigma_G^2\hat{a}^2\lambda^2/2} \\ e^{-A_B+A_B\exp(-b_{2\alpha}\hat{a}^2\lambda^2/2)-\sigma_G^2\hat{a}^2\lambda^2/2} \end{Bmatrix} d\lambda, \tag{6.4}$$

explicitly.

Let us use

$$\cos \lambda z = \sqrt{\pi z\lambda/2}\; J_{-1/2}(z\lambda) \, , \tag{6.5}$$

as with (5.6) in (5.5) ((5.7), (5.8)), to facilitate the integration. The result for (6.4) is explicitly

$$w_1(z)_{B-I} \simeq \sqrt{\frac{z}{2\pi}} \sum_{n=0}^{\infty} \int_{0-}^{\infty} \lambda^{+1/2} J_{-1/2}(z\lambda) \frac{(-1)^n \hat{a}^{n\alpha} A_B^n b_{1\alpha}^n}{n!} \lambda^{n\alpha} e^{-\Delta\sigma_G^2 \hat{a}^2 \lambda^2/2} d\lambda \tag{6.6a}$$

$$w_1(z)_{B-II} \simeq e^{-A_B} \sqrt{\frac{z}{2\pi}} \sum_{n=0}^{\infty} \frac{A_B^m}{m!} \int_{0-}^{\infty} \lambda^{1/2} J_{-1/2}(z\lambda) e^{-\hat{c}_{mB}^2 \lambda^2/2} d\lambda . \tag{6.6b}$$

Using (5.9) we find directly that,* in replacing z by zN_I in w_{1-BI}, we obtain explicitly

$$\left\{ \begin{array}{l} w_1(z)_{B-I} \equiv \hat{w}_1(\hat{z}) \simeq \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \hat{A}_\alpha^n}{n!} \Gamma\left(\frac{n\alpha+1}{2}\right) {}_1F_1\left(\frac{n\alpha+1}{2}; 1/2; -\hat{z}^2\right) , \\ (-\hat{z}_{oB} < z < \hat{z}_{oB}) \end{array} \right. \tag{6.7a}$$

$$\left\{ \begin{array}{l} \hat{z} \equiv zN_I/2\sqrt{2}G_B ; \\ \hat{z}_{oB} \equiv z_{oB}N_I/2\sqrt{2}\, G_B ; \end{array} \right. \tag{6.7a'}$$

and

$$\left\{ \begin{array}{l} w_1(z)_{B-II} \simeq e^{-A_B} \sum_{m=0}^{\infty} \frac{A_B^m}{m!} \frac{e^{-z^2/2\hat{c}_{mB}^2}}{\sqrt{2\pi\hat{c}_{mB}^2}} , z \le -z_{oB}, z \ge z_{oB} , \\ (z < -z_{oB}, z > z_{oB}) \end{array} \right. \tag{6.8a}$$

$$\hat{c}_{mB}^2 \equiv (m/\hat{A}_B + \Gamma_B')/(1+\Gamma_B') , \tag{6.8a'}$$

cf. (5.7); $\hat{A}_B = (2-\alpha)(4-\alpha)A_B$.

$$w_1(|z|) = 2w_1(z)_{B-II} , \quad z \ge 0 . \tag{6.8b'}$$

With large values of $|\hat{z}|$ ($< z_{oB}$), (6.7a) may be developed in the asymptotic

----

* Here we go from z to $z' = zN_I$; then from z' to $\hat{z} = z'/2\sqrt{2}G_B$, with a jacobian $= 2\sqrt{2}\, G_B$; this removes the factor $(2\sqrt{2}\, G_B)^{-1}$ in the evaluation of (6.6a).

series:

$$w_1(z)_{B-I} \equiv \hat{w}_1(\hat{z}) \simeq \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^n \hat{A}_\alpha^n}{n!\Gamma(-n\alpha/2)} \hat{z}^{-n\alpha-1}\left[1+\frac{(n\alpha+1)(\frac{n\alpha}{2}+1)+\ldots}{2\hat{z}^2}\right], \tag{6.9}$$

$$0 << |\hat{z}| \leq |\hat{z}_{oB}|.$$

These forms, $w_{1-BI,II}$, are, like their corresponding PD's, continuous at the "bend-over" point, $\pm z_{oB}$, cf. (5.22). Unlike the purely Class A interference, when $\Gamma_B' \to 0$ here there are no "gaps-in-time" $[G_B \to (4-\alpha)/4(2-\alpha)]$: this is a consequence of the fact that even if the system and external gauss noise component vanishes effectively ($\sigma_G^2 \to 0$), there still remains an inherent gaussian contribution $b_{2\alpha}A_B(>0)$, cf. (4.24d), stemming from the interaction between the relatively broad band incoming interference and the receiver's initial (linear) stages, e.g. $\Delta f_N > \Delta f_{ARI}$ (cf. comments pp. 74, 82 [Middleton, 1976]). Figures 6.1, 6.2 show $w_1(z)_B$, Eq. (6.3), based on the composite approximation (6.7), (6.8).

[We note, finally, that $\hat{w}_1(\hat{z})$, (6.7), is a proper pdf: $\hat{w}_1(\hat{z}) \geq 0$* and $\int_{-\infty}^{\infty} \hat{w}_1(\hat{z})d\hat{z} = 1$. We show here that this latter condition is satisfied. Let

$$I_\infty \equiv \lim_{b\to\infty} I(b);\ I(b) \equiv \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{A}_\alpha^n \Gamma\left(\frac{1+n\alpha}{2}\right) \int_0^b {}_1F_1\left(\frac{1+n\alpha}{2};1/2;-\hat{z}^2\right)d\hat{z}, \tag{6.10a}$$

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{A}_\alpha^n \Gamma\left(\frac{1+n\alpha}{2}\right) \int_0^{b^2} y^{-1/2} {}_1F_1\left(\frac{1+n\alpha}{2};\frac{1}{2};-y\right)dy, \tag{6.10b}$$

and then express ${}_1F_1$ in series form and integrate:

$$K_n(b^2) \equiv \int_0^{b^2} y^{-1/2} {}_1F_1\left(\frac{n\alpha+1}{2};1/2;-y\right)dy = \sum_{\ell=0}^{\infty} \frac{\left(\frac{n\alpha+1}{2}\right)_\ell (-1)^\ell}{(1/2)_\ell \ell!} \int_0^{b^2} y^{\ell-1/2} dy$$

$$= \sum_{\ell=0}^{\infty} \frac{\left(\frac{n\alpha+1}{2}\right)_\ell (-1)^\ell 2b^{2\ell+1}}{(1/2)_\ell \ell!\ (2\ell+1)}\ ; \tag{6.10c}$$

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* This follows from $|\hat{F}(ia\lambda)_I| \leq 1$, $\hat{F}_I \to 0$, $|\xi| \to \infty$, for the c.f.

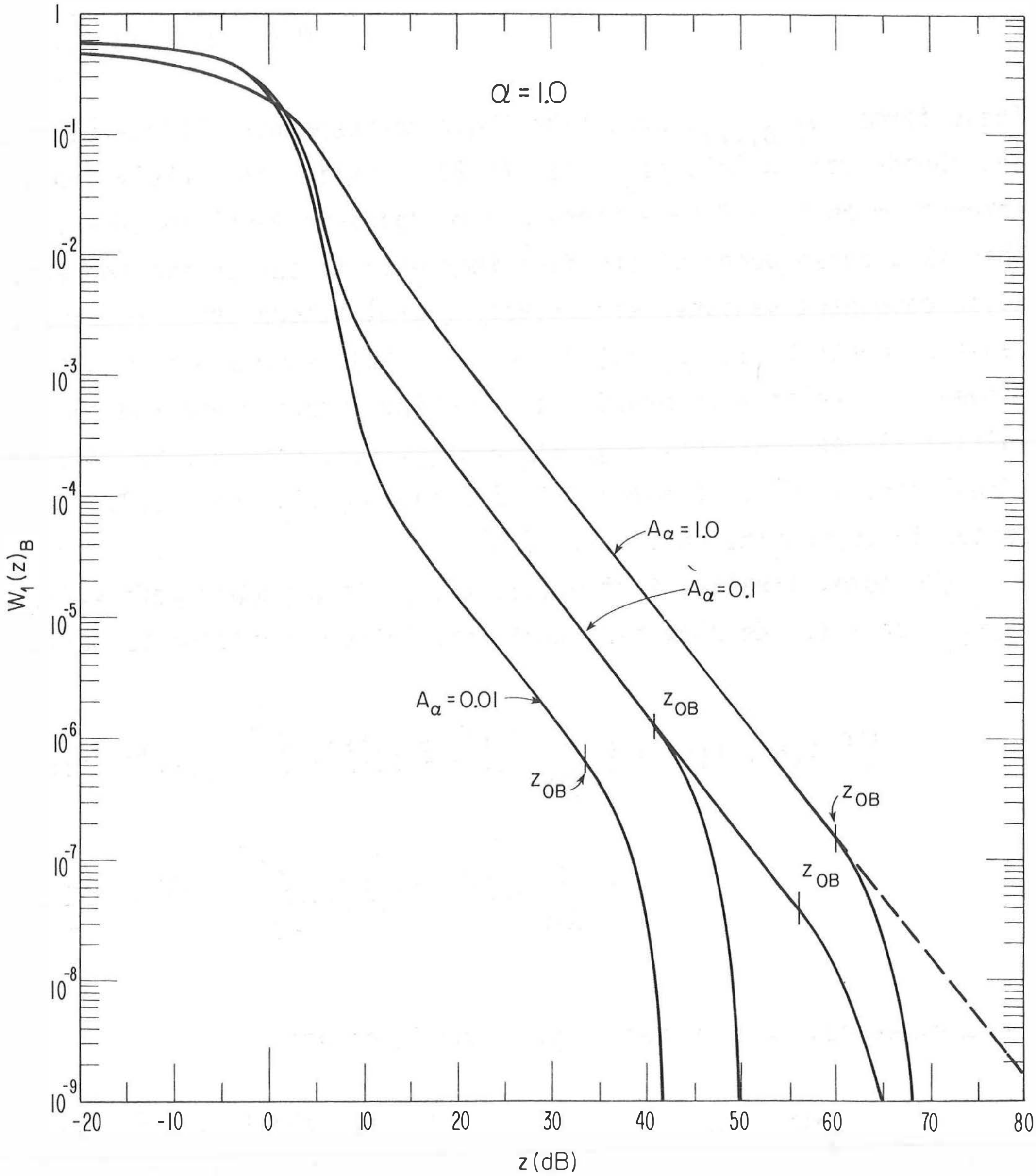

Figure 6.1 The (complete) pdf $w_1(z)_B$, (6.3), $0<z<\infty$ of the instantaneous amplitude for Class B interference, calculated from Eqs. (6.7a), (6.8a), for various A_α, given α, [See Fig. 5.6 for the associated PD and parameter values.]

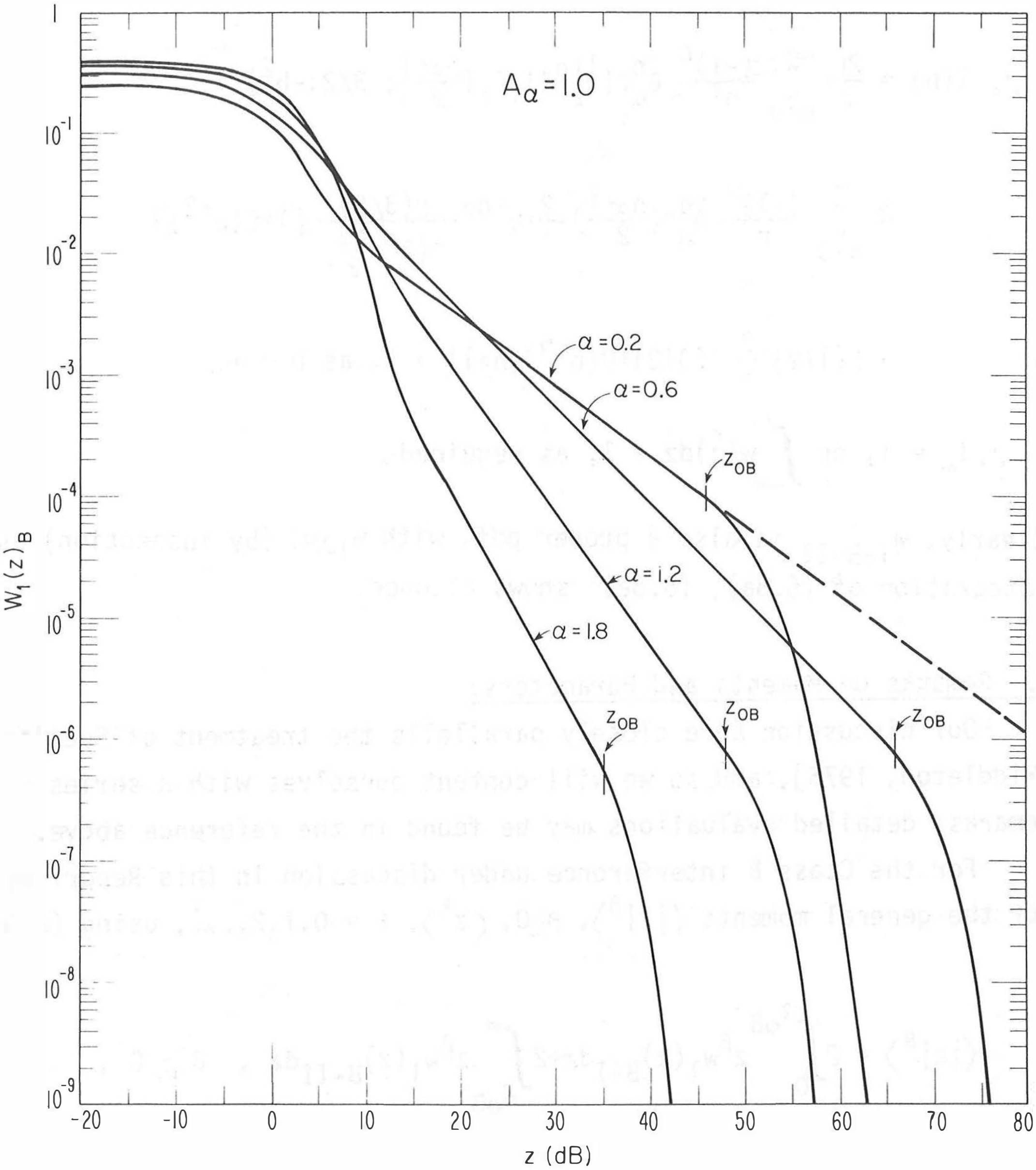

Figure 6.2 The (complete) pdf $w_1(z)_B$, (6.3), $0<z<\infty$, of the instantaneous amplitude for Class B interference, calculated from Eqs. (6.7a), (6.8a), for various α, given A_α. [See Fig. 5.7 for the associated PD and parameter values.]

Since $(2\ell+1)(1/2)_\ell = (3/2)_\ell$, we have

$$K_n(b^2) = 2b \sum_{\ell=0}^{\infty} \frac{(\frac{n\alpha+1}{2})_\ell(-1)^\ell b^{2\ell}}{(3/2)_\ell \ell!} = 2b \ {}_1F_1(\frac{n\alpha+1}{2};3/2;-b^2) \tag{6.10d}$$

$$\therefore I(b) = \frac{2b}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{A}_\alpha^n \Gamma(\frac{1+n\alpha}{2}) {}_1F_1(\frac{n\alpha+1}{2};\ 3/2;-b^2)$$

$$\cong \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{A}_\alpha^n \Gamma(\frac{n\alpha+1}{2}) \cdot \frac{2}{\pi} \cdot b^{-n\alpha} \frac{\Gamma(3/2)}{\Gamma(1-\frac{n\alpha}{2})} [1+0(b^{-2})]$$

$$\rightarrow \Gamma(1/2) \frac{2}{\pi} \Gamma(3/2)+0(b^{-n\alpha},n\geq 1) \rightarrow 1, \text{ as } b \rightarrow \infty. \tag{6.10e}$$

$\therefore I_\infty = 1$, or $\int_{-\infty}^{\infty} \hat{w}(\hat{z})d\hat{z} = 1$, as required.

Clearly, w_{1-B-II} is also a proper pdf, with $w_1 \geq 0$, (by inspection) as direct integration of (6.8a), (6.8a)' shows at once.

7. Remarks on Moments and Parameters:

Our discussion here closely parallells the treatment of Sections 5-7 [Middleton, 1976], and so we will content ourselves with a series of summary remarks; detailed evaluations may be found in the reference above.

For the Class B interference under discussion in this Report we may write for the general moments $\langle |z|^\beta \rangle$, $\beta \geq 0$, $\langle z^\ell \rangle$, $\ell = 0,1,2,\ldots$, using (6.3):

$$\langle |z|^\beta \rangle = 2\int_0^{z_{oB}} z^\beta w_1(z)_{B-I}dz + 2\int_{z_{oB}}^{\infty} z^\beta w_1(z)_{B-II}dz \ , \quad \beta \geq 0 \ , \tag{7.1}$$

where specifically we employ (6.7), (6.8) above for $w_{1-B-I,II}$. Similarly, we have for the integral moments

$$\langle z^\ell \rangle = \int_{-z_{oB}}^{z_{oB}} z^\ell w_1(z)_{B-I}dz + \left(\int_{z_{oB}}^{\infty} + \int_{-\infty}^{-z_{oB}}\right) z^\ell w_1(z)_{B-II}dz$$

$$= 2\int_0^{z_{oB}} z^{2\ell} w_1(z)_{B-I}dz + 2\int_{z_{oB}}^{\infty} z^{2\ell} w_1(z)_{B-II}dz \ ; \tag{7.2a}$$

$$\langle z^{\ell} \rangle = 0, \quad \ell = \text{odd}. \tag{7.2b}$$

As expected the odd moments vanish, since $w_1(z)$ is symmetrical about $z=0$. Furthermore, it is clear from the nature of $w_1(z)_{B-II}$, (6.8), that all (finite) (amplitude) moments of Class B interference exist, as required physically. [This was found to be the case for the envelope, as well [Middleton, 1976, p. 113].]

Specific moments may be determined by direct aplication of (6.7), (6.8) to (7.1), (7.2). In fact, exact expressions for $\langle z^{2\ell} \rangle$, (7.2a), may be simply found from the characteristic function, as we note below, Eqs. (7.6), so that we do not have to employ the approximate forms (6.7), (6.8) in (7.2). [A similar set of exact relations applies for the envelope moments $\langle \mathcal{E}^{2\ell} \rangle_B$, also.] Of course, if we want $\langle |z|^{\beta} \rangle$, then generally we must use (6.7), (6.8), in (7.1). Let us examine a few special cases of interest:

(i). $\langle |z|^{\beta} \rangle_I$: Here we have

$$\langle |z|^{\beta} \rangle_I \simeq 2\int_0^{z_{oB}(<\infty)} z^{\beta}(w_1(z)_{B-I}dz + \frac{2(-1)}{\Gamma(-\alpha/2)} \frac{\hat{A}_{\alpha}}{\sqrt{\pi}} \int_{\hat{z}_{oB}}^{\infty} \hat{z}^{\beta-\alpha-1}d\hat{z} \,, \tag{7.3}$$

from (6.9) in (7.1). The second integral is $O(\hat{z}^{\beta-\alpha})$: accordingly, only moments of order $\beta-\alpha<0$ or $(0\leq)\beta<\alpha$ can exist, based on $w_1(z)_{B-I}$. Since $(0<)\alpha<2$, this means that $\langle z^2 \rangle_I \to \infty$: $w_1(z)_{B-I}$ does not support a finite moment. (As noted earlier [Middleton, 1976, pp. 113-114] this is also the case for $\langle \mathcal{E}^2 \rangle_I$, for Class B envelopes.)

(ii). $\langle z^2 \rangle_{II}$: Here we calculate the second moment of z, based on $w_1(z)_{B-II}$, (6.8). Thus, we have

$$\langle z^2\rangle_{II} = e^{-A_B} \sum_{m=0}^{\infty} \frac{A_B^m}{m!} \int_{-\infty}^{\infty} \frac{z^2 e^{-z^2/2\hat{c}_{mB}^2}}{\sqrt{2\pi \hat{c}_{mB}^2}} \, dz$$

$$= e^{-A_B} \sum_{m=0}^{\infty} \frac{A_B^m \hat{c}_{mB}^2}{m!} = e^{-A_B} \sum_{m=0}^{\infty} \left(\frac{m}{\hat{A}_B} + \Gamma_B'\right) \frac{1}{m!(1+\Gamma_B')}$$

$$= \left(\frac{\frac{4-\alpha}{2-\alpha} + \Gamma_B'}{1+\Gamma'}\right) = 4G_B^2 \quad \text{, cf. (5.10), (6.8a)' ,} \tag{7.4}$$

which demonstrates the proper normalization factor ($=1/4G_B^2$), cf. (5.20) for Method 1. Accordingly, we have

$$\frac{\langle z^2\rangle_{II}}{4G_B^2} = 1 = \langle X^2\rangle_B / \Omega_{2B}(1+\Gamma_B') \ ; \therefore \ \langle X^2\rangle_B = \Omega_{2B}(1+\Gamma_B') \ . \tag{7.5}$$

Exact even moments for the instantaneous amplitudes of Class B interference are found at once from the results of Sec. 5.2A,B, and 5.3 [Middleton, 1976], when we note that

$$\langle z^{2\ell}\rangle_B = \lim_{z_0'\to\infty} (-1)^\ell \frac{d^{2\ell}}{d\lambda^{2\ell}} F_1(i\hat{a}\lambda|z_0')_{X:B+G} \ , \quad \xi = \hat{a}\lambda \ , \ (5.3) \ , \tag{7.6}$$

where $F_1(i\hat{a}\lambda|z_0')$ is given by (4.18) when the upper limit on the integral is replaced by z_0': see the comments of Sec. 5.2B [Middleton, 1976]. The results are directly

$$\langle z^0\rangle_B = 1 \text{ (as expected)} \tag{7.7a}$$

$$\langle z^2\rangle_B = 1 \text{ (as expected)} \tag{7.7b}$$

$$\langle z^4\rangle_B = \hat{a}^4[\tfrac{3}{2}\Omega_{4B}+3\Omega_{2B}^2(1+\Gamma_B')^2] = \frac{3}{2}\frac{\Omega_{4B}}{\Omega_{2B}^2(1+\Gamma_B')^2} + 3 \tag{7.7c}$$

$$\langle z^6\rangle_B = \hat{a}^6[\tfrac{5}{2}\Omega_{6B}+\frac{45}{2}\Omega_{4B}\Omega_{2B}(1+\Gamma')+15\Omega_{2B}^3(1+\Gamma')^3]$$

$$= \frac{5}{2}\frac{\Omega_{6B}}{\Omega_{2B}^3(1+\Gamma_B')^3} + \frac{45}{2}\frac{\Omega_{4B}}{\Omega_{2B}^2(1+\Gamma_B')^2} + 15, \text{ etc.}, \tag{7.7d}$$

where, as before,

$$(\Omega_{2\ell})_B \equiv A_B\langle\hat{B}_{oB}^{2\ell}\rangle/2^{\ell} \quad \text{, cf. (5.1a); and} \tag{7.8a}$$

$$\langle\ \rangle \equiv \left\langle\int_0^\infty (\ )dz\right\rangle_{A_o,e_{o\gamma},\mathcal{a}_{RT},\lambda} \text{, cf. (4.19a)} \tag{7.8b}$$

cf. (5.14a), [Middleton, 1976].

The odd moments, $\langle z^{2\ell+1}\rangle_B$, of course, vanish, by virtue of the symmetry of $w_1(z)_B$, or equivalently, the even (in ξ-) character of the (exact, and approximate) c.f.'s (4.18), (4.23). [We call the reader's attention, again, to the important conditions in the procedures for evaluating these (exact) Class B moments, and to the general canonical nature of the present analysis vis-à-vis the earlier work of Giordano and Haber [1970, 1972], and Furutsu and Ishida [1960], discussed on pp. 117-119 of [Middleton, 1976].

Finally, we have, as before [Middleton, 1976; Sec. 6], the global and generic parameters of the Class B, which are, as expected, the same as in the envelope cases, since we are dealing with the same interference at the receiver input. These parameters may be obtained, as indicated earlier, from envelope statistics [Sec. 6B, Middleton, 1976], or by analogous forms using the PD's and moments derived above [Sections 5,6]. These parameters are briefly described again in Section 2, preceeding. The practical conditions for Class B vs. Class A inteference models (vis-à-vis the receiver, of course) are just those already cited in Section 7 [Middleton, 1976] and will not be repeated here.

APPENDIX A: The Approximating PD's:

In order to use (4.5), (4.5b) with the appropriate approximating c.f.'s, $\hat{F}_1(ia\lambda)_{B-I,II}$, we must take care to employ the corresponding pdf's, $w_{1-B-I,II}$ in the correct fashion.* This is done as follows, assisted by Figure A-1, and remembering that $w_{I,II}$ are symmetrical pdf's (about z=0):

We first distinguish the following cases for the instantaneous amplitude:

$$\left.\begin{array}{ll} \text{A.-1.} & z_o < -z_{oB} \\ \text{A.-2.} & -z_{oB} < z_o < z_{oB} \\ \text{A.-3.} & z_{oB} < z_o \end{array}\right\} . \tag{A.1-1}$$

For the first, A-1., we have at once, by definition of P_1

$$P_1(z>z_{o1}) = \int_{z_{o1}}^{-z_{oB}} w_{II}dz + \int_{-z_{oB}}^{z_{oB}} w_I dz + \int_{z_{oB}}^{\infty} w_{II}dz \ . \tag{A.1-2}$$

which can be rewritten directly as

$$P_1(z>z_{o1}) = \int_{z_{o1}}^{-z_{oB}} w_{II}dz + \int_{-z_{oB}}^{z_{oB}} w_{II}dz + \int_{z_{oB}}^{\infty} w_{II}dz + \int_{-z_{oB}}^{z_{oB}} (w_I - w_{II})dz. \tag{A.1.3}$$

The final term in (A.1-3) is, with the aid of the symmetry properties of the pdf's,

* It is assumed here that $w_{B-I,II}$ $w_{1-B-I,II}$ are proper pdf's, i.e., $w_{I,II} \geq 0$ and $\int_{-\infty}^{\infty} w_{I,II} dz=1$. This is established in the text, cf. Sections 5.1,6(end).

$$\int_{-z_{oB}}^{z_{oB}}(w_I-w_{II})dz = 2\int_0^{z_{oB}}(w_I-w_{II})dz = \{1-P_1(|z|>|z_o|=z_{oB})_I\}$$

$$-\{1-P_1(|z|\geq|z_o|=-z_{oB})_{II}\}.$$

$$= P_1(|z|\geq|z_o|=z_{oB})_{II}-P_1(|z|\geq|z_o|=z_{oB}\ )_I \ , \qquad \text{(A.1-4)}$$

while the first three terms reduce at once to

$$\int_{z_{o1}}^{\infty} w_1(z)_{II}dz = P_1(z\geq z_{o1})_{II}. \qquad \text{(A.1-5)}$$

We next require that P_{1-I}, P_{1-II} be equal at $|z_o|=z_{oB}$: the respective PD's are continuous at the "turning point", $z_{oB}(>0)$. [Additional conditions for the composite approximation using the c.f.'s (or pdf's) for regions B-I,II are developed in Sec. (5.2).] Accordingly, we see from (A.1-3)-(A.1-5) that now

$$P_1(z>z_{o1}) = P_1(z>z_{o1})_{II} \quad ; z_{o1} < -z_{oB} \ , \qquad \text{(A.1-6)}$$

subject to the "joining" condition

$$P_1(|z|\geq z_{oB})_I = P_1(|z|\geq|z_{oB}|)_{II} \ , \qquad \text{(A.1-7)}$$

cf. (A.1-5) et seq.

We proceed in similar fashion for the cases A.1-2,3, to get

$$P_1(z>z_{o2,3}) = P_1(z>z_{o2,3})_I \ , \quad -z_{oB}<z_{o2,3}<z_{oB}, \qquad \text{(A.1-8)}$$

$$P_1(z>z_{o4}) = P_1(z>z_{o4})_{II} \quad , \quad z_{o4} > z_{oB} \qquad \text{(A.1-9)}$$

A similar procedure for the modulus, $|z|$, (shaded regions in Fig. A.1) may be used to give us directly

$$P_1(|z|>|z_0|) = P_1(|z|\geq|z_0|)_I \ , \ 0 \leq |z_0| \leq z_{0B}, \qquad (A.1-10)$$

$$= P_1(|z| \geq |z_0|)_{II} \ , \ |z_0|>z_{0B} \ (>0) \qquad (A.1-11)$$

Specifically, we can write

$$P_1(|z|>|z_0|)_{I,II} = P_1(|z|>|z_0|)_{B-I,II} = 1-2\int_0^{|z_0|} w_1(z)_{B-I,II} dz \ , \qquad (A.1-12)$$

with the appropriate domains $0\leq|z_0|<z_{0B}$, $|z_0|>z_{0B}$, respectively. Hence follow Equations (5.5a,b) in the main text. (Similarly, if we replace z by $\mathcal{E}$, z_{0B} by $\mathcal{E}_B$, etc., Eqs. (A.1-10-12) verify our earlier results, Eqs. (2.7), [Middleton, 1976,1977].)

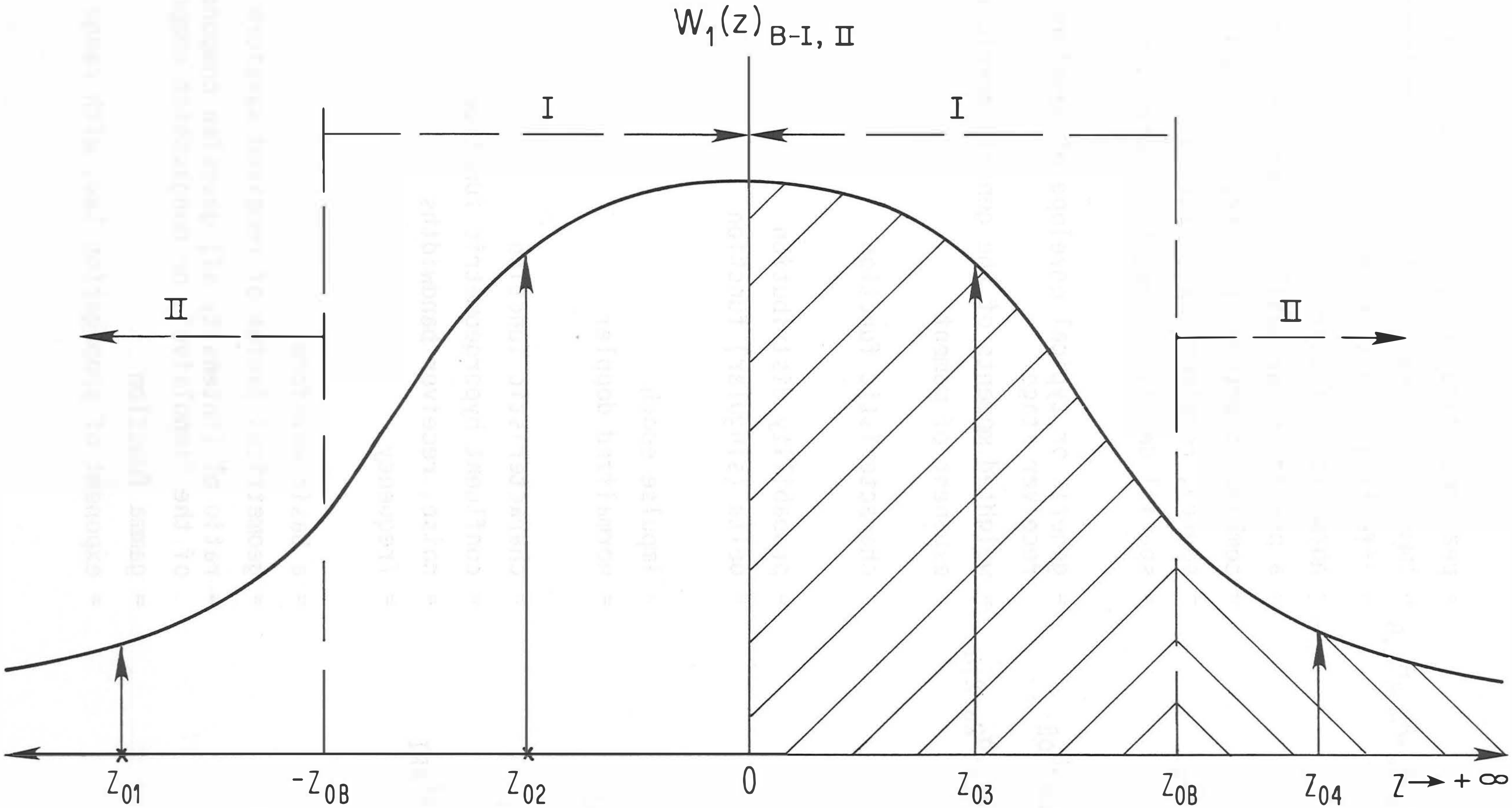

Figure A-1 Schema of the pdf's, $w_1(z)_{B-I,II}$, appropriate to the regions B-I,II.

Glossary of Principal Symbols

A.	A_o	= peak amplitude of typical input signal
	$A_A, A_B, A_\infty, A_\infty, A^A_{\infty,B}$	= Impulsive Indexes, (Class A,B interference)
	A_α	= effective Impulsive Index
	$\hat{a}$	= normalizing factor
	APD	= a posteriori probability; here 1-Distribution=P_1
	ARI	= combined aperture-IF-IF receiver input stages
	$\mathcal{a}_T, \mathcal{a}_R$	= source, receiver beam patterns
	α	= spatial density propagation parameter
B.	$B_o, \hat{B}_{oA}, \hat{B}_{oB}$	= generic or typical envelope of waveform from ARI receiver stage
	$b_{1\alpha}, b_{2\alpha}, b_{2\alpha+2\|\alpha}$	= weighted moments of the generic envelope B_{oB}
	β	= exponent of moment
C.	c.f.	= characteristic function
D.	D_1	= probability distribution
	δ	= delta (singular) function
E.	$\hat{\varepsilon}$	= impulse epoch
	$\varepsilon_o, \varepsilon_d$	= normalized doppler
F.	$\hat{F}_1, F_1$	= characteristic function
	${}_1F_1$	= confluent hypergeometric function
	$\Delta f_N, \Delta f_{ARI}$	= noise, receiver bandwidths
	f	= frequency
G.	G_o	= a basic waveform
	$g(\underset{\sim}{\lambda})$	= geometrical factor of received waveform
	Γ'_B	= ratio of (intensity of) gaussian component to that of the "impulsive", or nongaussian component
	$\Gamma(x)$	= gamma function
	γ	= exponent of propagation law, with range
H.		

I. $\hat{I}_T, \hat{I}_\infty$ = exponent of characteristic function
I_c = incomplete Γ-function
$\hat{i}_R$ = unit vector

J. J_o, J_1 = Bessel function, 1st-kind, (0,1 order).
J_Λ = jacobian

K. ξ = c.f. variable

L. Λ = domain of integration
λ = argument of the c.f.
$\underset{\sim}{\lambda}$ = (λ, θ, ϕ), coördinates

M. μ = exponent of source density law with range
μ_d = normalized doppler

N. n.b. = narrow-band

O. Ω_{2B} = mean intensity of the nongaussian component
ω, ω_o = angular frequencies (ω_o= carrier angular fr.)

P. P_1 = APD or Exceedance Probability
pdf = probability density function
Ψ, ϕ = phase of narrow band wave
ϕ_T, ϕ_R = aperture phase

Q.

R. r = c.f. variable
ρ = poisson "density"

S. $\sigma, \sigma_G, \hat{\sigma}, \Delta\sigma_G^2, \sigma_\Lambda, \sigma_R^2, \sigma_E^2$ = variances
$\sigma_{S,V}$ = source density

T. $T_s, \bar{T}_{s:B}$ = emission duration
t, t_1, t_2 = times
$\underset{\sim}{\theta}, \underset{\sim}{\theta}'$ = sets of waveform parameters

T.	Θ	= error function
U.	U, U_{nb}	= basic waveforms out of ARI receiver stage
	u_o, u_B	= normalized envelope waveform at output of ARI stages
V.		
W.	w_1, W_1	= probability density function
X.	X	= instantaneous amplitude
	x_o	= a c.f. variable
Y.		
Z.	z	= (normalized) instantaneous amplitude
	z_o	= a normalized time, also, a normalized amplitude threshold
	z_{oB}	= (normalized) "bend-over" point

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Statistical-Physical Models of Man-Made and Natural Radio Noise Part III: First-Order Probability Models of the Instantaneous Amplitude of Class B interference

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15. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

This Report is the third ("Part III") in a continuing series devoted to the development of analytically tractable, statistical-physical models of man-made and natural electromagnetic interference. Here, the first-order statistical probability densities (pdf's) and the associated exceedance probabilities (PD's, or APD's) are obtained for the instantaneous amplitudes (X), and instantaneous magnitudes, |X|, of Class B noise. These are needed not only for experimental studies but, also, particularly for the analysis and evaluation of the performance of optimum and suboptimum receivers in Class B interference environments.

As in the earlier studies of the envelope statistics of Class B noise [Middleton, 1976], a two-function approximation is needed for the characteristic function and hence for the corresponding pdf's and PD's. Two methods of determining the six (basic) parameters which describe these first-order statistics and thus joining the approximate forms (pdf's and PD's) are outlined. Method 1 is approximate, was used earlier [Middleton, 1976, 1977], and has the advantage of

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somewhat greater computational simplicity, with the disadvantage, however, of yielding too low values of the PD at low values of the argument (X), when the gaussian component is small. Method 2 is "exact", and somewhat more complex computationally. The joining process involved in both methods has been essentially described earlier [Middleton, 1976, 1977] but is developed further here. The basic parameters are, in any case, the same as those derived for the envelope statistics. The excellent agreement with experiment observed for the envelope data accordingly applies here, as well.

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