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Statistical-Physical Models of Man-Made and Natural Radio Noise

PART IV: DETERMINATION OF THE FIRST-ORDER PARAMETERS OF CLASS A AND B INTERFERENCE

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Statistical-Physical Models of Man-Made and Natural Radio Noise

PART IV: DETERMINATION OF THE FIRST-ORDER PARAMETERS OF CLASS A AND B INTERFERENCE

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PREFACE

This Report is the fourth (i.e., Part IV) in a series of ongoing studies [Middleton, 1974, 1976, 1978] of the general electromagnetic (EM) interference environment arising from man-made and natural EM noise sources, and is also part of the continuing analytical and experimental effort whose general aims are [Spaulding and Middleton, 1975, 1979]:

(1). to provide quantitative, statistical descriptions of man-made and natural electromagnetic interference (as in this series);

(2). to indicate and to guide experiment, not only to obtain pertinent data for urban and other EM environments, but also to generate standard procedures and techniques for assessing such environments;

(3). to determine and predict system performance in these general electromagnetic milieux, and to obtain and evaluate optimal system structures therein, for

  (a). the general purposes of spectrum management;

  (b). the establishment of appropriate data bases thereto; and

  (c). the analysis and evaluation of large-scale telecommunication systems.

With the aid of (1) and (2) one can predict the interference characteristics of selected regions of the electromagnetic spectrum, and with the results of (3), rational criteria of performance can be developed to predict the successful or unsuccessful operation of telecommunication links and systems in various classes of interference. Thus, the combination of the results of (1)-(3) provide specific, quantitative procedures for spectral management, and a reliable technical base for the choice and implementation of policy decisions thereto.

The man-made EM environment, and most natural EM noise sources as well, are basically "impulsive", in the sense that the emitted waveforms have a highly structured character, with significant probabilities of large interference levels. This is noticeably different from the usual normal (gaussian) noise processes inherent in transmitting and receiving elements. This highly structured character of the interference can drastically degrade performance of conventional systems, which are optimized, i.e., designed to operate most effectively, against the customarily assumed normal background noise processes. The present Report is devoted to the problems of (1), (2) above, namely, to provide adequate statistical physical models, verified by experiment, of these general "impulsive", highly non-gaussian interference processes, which constitute a principal corpus of the interference environment, and which are required in the successful pursuit of (3), as well. The principal new results here are:

(i). Analytical and empirical methods for determining the (first-order) parameters of both the approximate and exact Class A and B noise models;*

(ii). Procedures for measuring the accuracy of the sample estimates of these model parameters.

Finally, we emphasize, again, that it is the quantitative interplay between the experimentally established,** analytical model-building for the electromagnetic environment, and the evaluation of system performance therein, which provides essential tools for prediction and performance, for the development of adequate, appropriate data bases, procedures for effective standardizations, and spectrum assessment, required for the effective management of the spectral-use environment.

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* Class A and Class B noise are distinguished, qualitatively, by having input bandwidths which are respectively narrower and broader than that of the (linear) front-end stages of the typical (narrow-band) receiver in use. More precise definitions are developed in the text following.

** Excellent experimental corroboration has been achieved, on the basis of envelope data for both the Class A and B interference processes [cf. Middleton, 1976, Sec. 2.4]. An equivalent corroboration for the corresponding amplitude data is accordingly inferred.

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ABSTRACT

Methods of determining the first-order parameters of both the approximate and exact Class A and B noise models are derived, for both the ideal case of infinite sample data and the practical cases of finite data samples. It is shown that all (first-order) parameters of these models can, in principle, be obtained, exactly or approximately, from the ideal or practical measurements. [All first-order parameters of Class A but only the (first-order) even moments of the Class B models are exactly obtainable in the ideal case.] Procedures for establishing meaningful measures of the accuracy of the parameter estimates in the practical cases are identified: suitably adjusted, non parametric, small-sample tests of "goodness-of-fit" (such as Kolmogorov-Smirnov tests) provide the principal techniques for establishing accuracy, at an appropriately selected significance level (α_0). The roles of robustness and stability of the parameters and their estimators are also discussed.

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STATISTICAL-PHYSICAL MODELS OF MAN-MADE AND NATURAL RADIO NOISE

Part IV: Determination of the First-Order Parameters of Class A and B Interference

Section 1. Introduction:

In various recent [Middleton,[1]-[3]; 1977,1976,1974] and ongoing studies [Middleton,[4],1978,[5],1978; Spaulding and Middleton, 1977] statistical-physical models of man-made and natural electromagnetic (EM) interference have been developed which are both analytically tractable and in excellent agreement with experiment for a broad range of practical examples. The specific statistics obtained so far are first-order probability distributions (PD's) and densities (pdf's), associated moments, and the pertinent parameters needed to specify the approximating forms of these PD's and pdf's. The principal interference models are the so-called Class A and Class B types [Middleton,[1], 1977] respectively distinguished by input spectral bandwidths (Δf_N) smaller than or larger than the bandwidth (Δf_{ARI}) of a typical receiver's (linear) front-end stages. The approximating Class A noise models are described by a single first-order characteristic function (c.f.) with three parameters, whereas the Class B types require two, suitably connected, c.f.'s (or corresponding pdf's or PD's) and six parameters. [We stress that these parameters are not at all *ad hoc*, but are derived from the underlying physical model, and are themselves experimental observables, as noted earlier [Middleton, 1977, 1976], and as discussed further below.]

Basic problems in applying Class A and B models to experimental observation are determining parameter values, and additionally in Class B cases, effecting a suitable "joining" of the approximating pdf's (or PD's). Accordingly, the purpose of this Report is threefold: (i), to indicate an improved method (vis-à-vis Sec. 3.2.1 of [Middleton,[1], 1977]) of achieving the "joining" of the approximating pdf's (or PD's) for Class B models; (ii), to outline effective formal procedures for determining not only the model parameters for the (first-order) approximate forms, but all the parameters associated with the corresponding exact first-order characteristic functions (pdf's, etc.); and (iii), to indicate in preliminary fashion useful and necessary procedures for estimating these parameter values for the limited data samples available in practice. In this purely analytic report our interest is to initiate a technical basis for subsequent practical applications to both experimental model studies and the measurement of EM interference environments generally. In addition, of course, these models play a critical rôle in determining the performance of optimum and suboptimum receiving systems in such realistic environments [Spaulding and Middleton, 1975,1977,1979].

Finally, this study is organized as follows: Section 2 describes the new, exact "joining" process generally and in some analytic detail for Class B noise; Sections 3 and 4 present formal determinations of Class A and B model parameters, respectively; while Section 5 is devoted to identifying procedures for determining the accuracy of the parameter estimates when finite samples are used. Section 5 also surveys some of the empirical statistical considerations imposed by the finite data sample sizes available and the possibility of instabilities in the underlying mechanisms generating the data themselves. [Only the pertinent envelope statistics are considered here (for the most part), since it is envelope data which are conveniently obtained, and since the parameters of the envelope and corresponding instantaneous amplitude distributions are necessarily the same.] The paper concludes with a short summary of results and some important next steps in extending theory and applications. [Section 6].

Section 2. Remarks on the Joining Procedure for Class B Noise Models:

Because the exact characteristic function (c.f.) for Class B interference [cf. (2.38), [Middleton [1], 1977]; (2.87), [Middleton, 1976]] must be approximated, with two distinct c.f.'s, $\hat{F}_{1-I}, \hat{F}_{1-II}$ [cf. Sec. 2.6.1, [Middleton [1], 1977]], it is essential that the resulting two pdf's, $w_{1-I,II}$, be suitably connected or joined, at some appropriate (normalized[1]) envelope

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1. It is convenient analytically to use the normalized forms of the envelope E: $\mathcal{E} = Ea_{(A)}, Ea_B$, where a_A, a_B are given respectively in (3.2b), (4.1a). Thus, also, $\mathcal{E}_0 = E_0 a_{A,B}$, $\mathcal{E}_B = E_B a_B$, etc., in the following; cf. Section III, [Middleton, 1977].

value, $\mathcal{E}_B$, as shown in Fig. 2.1. This value, in turn, is determined by the conditions placed upon these pdf's at $\mathcal{E}_B$, as we show in the following:

The conditions on $w_{1-I,II}$ at $\mathcal{E} = \mathcal{E}_B$ [and correspondingly upon the PD's, $P_{1-I,II}$], which we choose are:*

(2.1)

(i). No "mass-points" or delta functions in $w_{1-I,II}$ at $\mathcal{E} = \mathcal{E}_B$; viz., $P_{1-I} = P_{1-II} = P_1$ are continuous at $\mathcal{E} = \mathcal{E}_B$;

(ii). The pdf's are continuous at $\mathcal{E}_B$; viz., $w_{1-I} = w_{1-II} = w_1$, or $dP_{1-I}/d\mathcal{E}_B = dP_{1-II}/d\mathcal{E}_B$ $(\doteq dP_1/d\mathcal{E}_B)$: the derivatives of the PD's are continuous at $\mathcal{E}_B$;

(iii). The pdf's are "smooth" at $\mathcal{E}_B$, viz. $dw_{1-I}/d\mathcal{E}_B = dw_{1-II}/d\mathcal{E}_B$ $(\doteq dw_1/d\mathcal{E}_B)$, or $d^2P_{1-I}/d\mathcal{E}_B^2 = d^2P_{1-II}/d\mathcal{E}_B^2$ $(\doteq d^2P_1/d\mathcal{E}_B^2)$: the second derivatives of the PD's are likewise continuous at $\mathcal{E}_B$.

In fact, $\mathcal{E}_B$ is the "turning point" or point of inflexion $(d^2P_1/d\mathcal{E}_B^2 = 0)$ of the exact PD, P_1, and is always estimated from the empirical PD, or exceedance probability $P_{1-expt'l.}$, cf. Fig. 3.5 (II), [Middleton, 1976, 1977], and as shown in Fig. 2.2 here. Condition (i), (2.1), insures that the approximating PD is P_{1-I} for $\mathcal{E} \leq \mathcal{E}_B$ and P_{1-II} for $\mathcal{E} \geq \mathcal{E}_B$. Conditions (ii), (iii) insure the mutual continuity and smoothness of the associated pdf's at the joining point $\mathcal{E}_B$. Continuity and smoothness of the approximating pdf's, $w_{1-I,II}$, elsewhere, is insured by their respective characteristic functions [c.f. Secs. 3, 4, below. [Conditions (i)-(iii) also provide six analytic relations* which may be used in determining the six global (or generic) parameters of the approximating PD's (and pdf's and c.f.'s), cf. Sec. 4 ff.]

The stated consequences of Condition (i) are easily demonstrated: let us consider some threshold $(0 \leq \mathcal{E}_{01} \leq \mathcal{E}_B)$ and calculate P_1 using the approximate pdf's. From Fig. 2.1 it is at once evident that

$$P_1(\mathcal{E}>\mathcal{E}_{01}) \doteq \int_{\mathcal{E}_{01}}^{\mathcal{E}_B} w_1(\mathcal{E})_I d\mathcal{E} + \int_{\mathcal{E}_B}^{\infty} w_1(\mathcal{E})_{II} d\mathcal{E} , \qquad (2.2)$$

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* See, however, Eq. (4.15), and the discussion leading to (4.12)-(4.15); also Eq. (4.18).

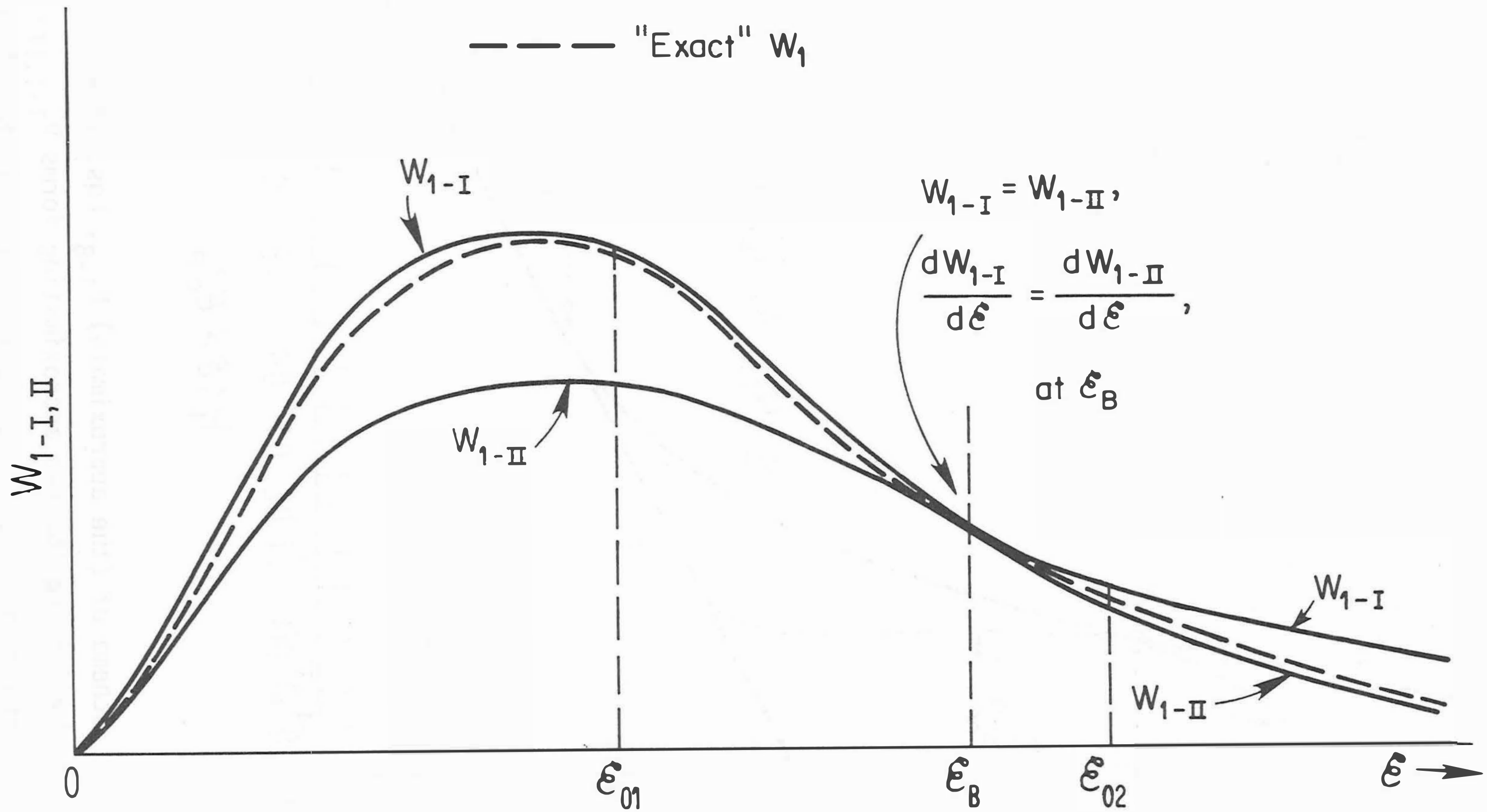

Figure 2.1 Schema of Condition(s) (i)-(iii), Eq. (2.1), for joining the approximating pdf's $w_{1-I,II}$, according to Eqs. (2.4), (2.5), for the Class B noise model.

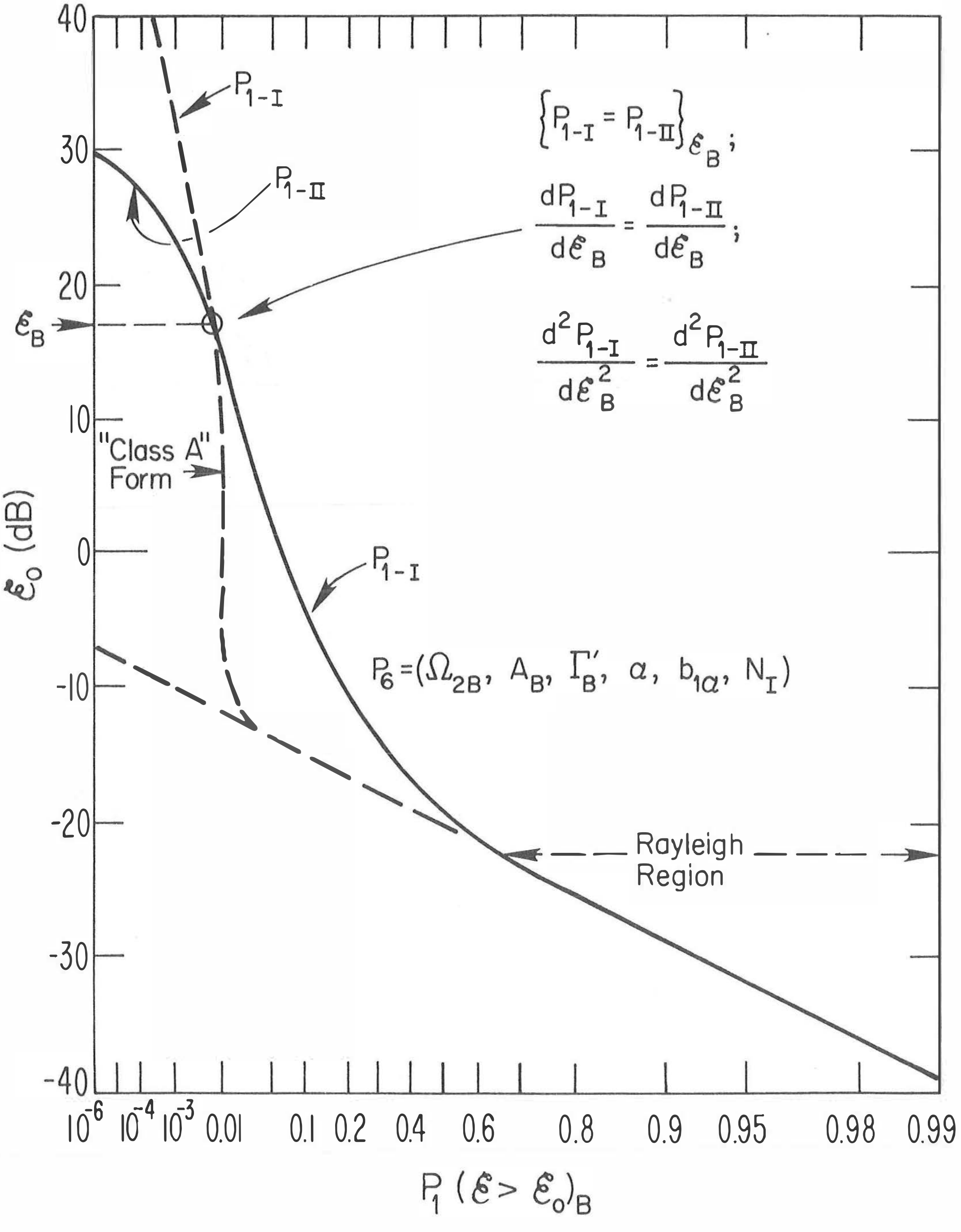

Figure 2.2 Schema of (the approximate) P_{1-B}, Eqs. (2.4), (2.5), obtained by joining the two approximating forms $P_{1-I,II}$, according to the conditions of Eq. (2.1). [This is the PD form of Fig. 2.1.]

for which the right member is equivalently written

$$\int_{\mathcal{E}_{01}}^{\mathcal{E}_B} w_{1-I}\, d\mathcal{E} + \int_{\mathcal{E}_B}^{\infty} w_{1-I}\, d\mathcal{E} + \int_{\mathcal{E}_B}^{\infty} (w_{1-II} - w_{1-I})\, d\mathcal{E}, \tag{2.2a}$$

or

$$P_1(\mathcal{E} \geq \mathcal{E}_{01}) \doteq P_1(\mathcal{E} \geq \mathcal{E}_{01})_I + \{P_1(\mathcal{E} \geq \mathcal{E}_B)_{II} - P_1(\mathcal{E} \geq \mathcal{E}_B)_I\}, \tag{2.3}$$

But, from Condition (i), (2.1), it follows at once that

$$\left.\begin{aligned} P_1(\mathcal{E} \geq \mathcal{E}_{01}) &\doteq P_1(\mathcal{E} \geq \mathcal{E}_{01})_I \\ w_1(\mathcal{E}) &\doteq w_1(\mathcal{E})_I \end{aligned}\right\} , \; 0 \leq \mathcal{E}_{01} \leq \mathcal{E}_B. \tag{2.4a}$$

Similarly, when $\mathcal{E}_{02}$ $(\geq \mathcal{E}_B)$, we have directly

$$\left.\begin{aligned} P_1(\mathcal{E} \geq \mathcal{E}_{02}) &\doteq P_1(\mathcal{E} \geq \mathcal{E}_{02})_{II} \\ w_1(\mathcal{E}) &\doteq w_1(\mathcal{E})_{II} \end{aligned}\right\} , \; \mathcal{E}_B \leq \mathcal{E}_{02} \, . \tag{2.5}$$

cf. Figs. (2.1) . (2.2), Equations (2.4), (2.5) exhibit the desired result, employed in the previous analyses, cf. [Middleton, 1976, 1977, Refs. [1], [2], [4]], that when the threshold $\mathcal{E}_0$ lies below the "turning point", $\mathcal{E}_B$ we must use the type I approximation $(\hat{F}_{1-I} \rightarrow w_{1-I}, P_{1-I})$, and when $\mathcal{E}_0$ falls above $\mathcal{E}_B$, the type II approximation $(\hat{F}_{1-II} \rightarrow w_{1-II}, P_{1-II})$.

In order to proceed further with the determination of (first-order) Class B model parameters, it is necessary first to consider the corresponding problems for Class A models, which are of concern here in their own right, and which provide structures and procedures needed in the Class B cases.

Section 3. Formal Determination of Class A Model Parameters:

We begin by citing the exact characteristic functions (c.f.'s), determined earlier ([Middleton, 1974,1976,1977-Refs. [1]-[3]]) and the (exact) expression for the even-order moments (of the envelope). From this we proceed to the formal determination of the three (first-order) parameters of the approximate analytical model, and all the (first-order) parameters of the exact model. These results, in turn, may be directly applied to the actual estimatio of the parameters from limited experimental data, as initiated in Section 5 following.

The exact first-order c.f. for Class A interference, with an independent gaussian component, may be expressed in two principal forms:

Eq. (2.50), [Middleton, 1976]

$$\hat{F}_1(ia\lambda)_{A+G} = \exp\{-A_A-\sigma_G^2 a_A^2\lambda^2/2+A_A\left\langle\int_0^{z_0(<\infty)} J_0(a_A\lambda\hat{B}_{0A})dz\right\rangle_{z_0,\lambda,\theta} \tag{3.1a}$$

or

(I):

$$\boxed{\hat{F}_1(ia\lambda)_{A+G} = \exp\{-\Omega_{2A}(1+\Gamma_A')a_A^2\lambda^2/2+\sum_{k=2}^{\infty}\frac{(a_A\lambda)^{2k}(-1)^k}{2^k(k!)^2}\Omega_{2k,A}\}} \ , \tag{3.1b}$$

obtained on expanding J_0 in (3.1a), where specifically [(2.75d), [Middleton, 1976]],

$$\Omega_{2k,A} \equiv A_A\langle\hat{B}_{0A}^{2k}\rangle/2^k, \quad k \geq 1; \quad \hat{B}_{0A}^{2k} = \left\langle\int_0^{z_0}\hat{B}_{0A}^{2k}(z,\theta',\lambda)dz\right\rangle_{z_0,(\lambda,\theta'\equiv\theta)}, \tag{3.2a}$$

and

$$a_A^2 = \{2\Omega_{2A}(1+\Gamma_A')\}^{-1} \quad ; \ \Gamma_A' \equiv \sigma_G^2/\Omega_{2A} \ , \tag{3.2b}$$

where σ_G^2, as before, is the intensity of the gaussian component, and Ω_{2A} is the mean intensity of the nongaussian or "impulsive" component, of the Class A noise model; A_A is the Impulsive Index, cf. (2.16), (2.18), et seq. [1], which is a key measure of the non-normal character of the interference

[Middleton, 1977, 1976]. The second, equivalent form of the exact c.f., (3.1a), from which the required approximate form (3.6) below is obtained, follows from {(2.76) in (2.69), (2.59), in (2.50), [Middleton, 1976]}, and is

(II):

$$\hat{F}_1(ia\lambda)_{A+G} = \exp\{-A_A-\sigma_G^2 a_A^2\lambda^2/2+A_A e^{-a_A^2\lambda^2\Omega_{2A}/2A_A} \cdot\left[1+\sum_{\ell=2}^{\infty}\frac{(\Omega_{2A}/A_A)^{\ell}(-1)^{\ell}}{2^{\ell}(\ell!)^2}\hat{C}_{2\ell}\cdot(a\lambda)^{2\ell}\right]\} \tag{3.3}$$

with

$$\hat{C}_{2\ell} \equiv \ell!(-1)^{\ell}\left\langle\int_0^{z_0} {}_1F_1(-\ell;1;[A_A\hat{B}_{0A}^2/2]/\Omega_{2A})dz\right\rangle_{z_0,\underset{\sim}{\theta}'}, \tag{3.3a}$$

cf. (2.75), (2.75a-d), [Middleton, 1976].

The exact forms of the even moments of the (normalized) envelope are obtained from (3.1b) in

$$\langle\mathcal{E}_A^{2k}\rangle = \frac{(k!)^2 2^{2k}}{(2k)!}\left[(-1)^k\frac{d^{2k}}{d\lambda^{2k}}\hat{F}_1(ia\lambda)\right]\Bigg|_{\lambda=0}, \tag{3.4}$$

cf. (5.10a), [Middleton, 1976], and Sections 5.2a, [Middleton, 1976] generally. The results are specifically

$$\left.\begin{aligned} &\langle\mathcal{E}_A^{(0)}\rangle = 1 \quad ; \quad \langle\mathcal{E}_A^{2}\rangle = 1\,, \\ &\langle\mathcal{E}_A^{4}\rangle = \frac{\Omega_{4A}}{\Omega_{2A}^2(1+\Gamma_A')^2}+2\,, \\ &\langle\mathcal{E}^{6}\rangle = \frac{\Omega_{6A}}{\Omega_{2A}^3(1+\Gamma_A')^3}+\frac{9\Omega_{4A}}{\Omega_{2A}^2(1+\Gamma_A')^2}+6\,,\ \text{etc.} \end{aligned}\right\} \tag{3.5}$$

with $\mathcal{E}_A = Ea_A$.

3.1 Exact Parameter Determination (Class A):

We start by indicating a procedure for determining the three basic first-order parameters $(\Omega_{2A}, A_A, \Gamma'_A)$ of both the approximate and exact Class A models. We then go on to show how all the parameters of the exact (first-order) model can be found in principle. Here, as in Sec. 4, we for the time-being assume that we are dealing, both "experimentally" and analytically, with the infinite-population (or "ideal-experimental") and limiting theoretical distributions and their parameters. Later, in Section 5 below, we shall briefly consider how to treat the practical cases of finite data samples and estimates of the theoretical or infinite population parameters, PD's and pdf's.

The desired approximate Class A model is obtained from the exact c.f., (3.3), by omitting the terms $\ell \geq 2$. The result is [cf. (3.2), Middleton, 1977], (3.5), [Middleton, 1976] the expanded version[2]

$$\hat{F}_1(ia\lambda)_{A+G} \doteq e^{-A_A} \sum_{m=0}^{\infty} \frac{A_A^m}{m!} e^{-\hat{\sigma}_{mA}^2 \lambda^2/2} \quad ; \quad 2\hat{\sigma}_{mA}^2 = \left(\frac{m}{A_A} + \Gamma'_A\right)/(1+\Gamma'_A) . \tag{3.6}$$

The three (global)-parameters appearing in this approximation are $(\Omega_{2A}, A_A, \Gamma'_A)$, cf. (3.2). The associated PD is from (3.3), [Middleton, 1977]:

$$P_1(\mathcal{E} \geq \mathcal{E}_0)_A \cong e^{-A_A} \sum_{m=0}^{\infty} \frac{A_A^m}{m!} e^{-\mathcal{E}_0^2/2\hat{\sigma}_{mA}^2} \quad , \quad \mathcal{E}_0 = E_0 a_A . \tag{3.7}$$

(The associated pdf is given by Eq. (2.3), [Middleton, 1977], Eq. (4.2), [Middleton, 1976]. The PD, (3.7), and pdf, here are a proper PD and pdf, cf. footnote 8.]

To determine the three parameters we must "fit" in some sense (3.7) to the (postulated) given "ideal-experimental" data for P_{1-A}. An apparent difficulty, however, is that P_{1-A}, (3.7), is an approximate form for all $(\mathcal{E}_0 > 0, < \infty)$,

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2. The a^2 in Eqs. (2.1), (3.2), [Middleton, 1977] should be omitted.

while the data are infinite-population, or "exact" data. This difficulty can be overcome by noting that for small $(\mathcal{E}_0, \mathcal{E})$ the form of the PD (and pdf) is governed principally by the behaviour of the c.f. as λ becomes large, which means that for the nongaussian component process (A) the exponential term in the exponent of (3.3) is controlling here (as well as for $\lambda \to 0$, for the large $\mathcal{E}$, $\mathcal{E}_0$). But this is the c.f. (3.6), which is now the exact c.f. as $(\mathcal{E}, \mathcal{E}_0 \to 0)$, and which correspondingly leads to the exact PD, (3.7), as $\mathcal{E}_0 \to 0$. [Of course, for $(\mathcal{E}_0, \mathcal{E})$ at intermediate values, (3.6), (3.7), and pdf, are approximate, albeit excellent approximations, as the data of Figs. 2.1, 2.2, [Middleton, 1976] indicate, for example. These analytical forms become exact again as $\lambda \to 0$ or $(\mathcal{E}, \mathcal{E}_0 \to \infty)$, also.] Consequently, this suggests that we employ the limiting properties of the (ideal) data sample along with the analytic forms (3.7) as $\mathcal{E}_0 \to 0$. These are: (1), the PD, and (2), $dP/d\mathcal{E}_0^2$, as $\mathcal{E}_0 \to 0$. The required third relation is the expression for the (exact) second moment[3], cf. (3.5), $\langle E_A^2 \rangle = \langle \mathcal{E}_A^2 \rangle / a_A^2 = 2\Omega_{2A}(1+\Gamma_A')$. Accordingly, the needed three relations for determining the infinite-population parameters $(\Omega_{2A}, A_A, \Gamma_A')$ are:

(i). 2nd moment: $\langle E_A^2 \rangle_{\text{ideal-xpt.}} \equiv \langle E_A^2 \rangle = 2\Omega_{2A}(1+\Gamma_A')$ (analytic) (3.8a)

(ii). PD at limitingly small thresholds:[4] $[P_1 |_{\mathcal{E}_0^2 \to 0}]_{\text{i-expt}} = (P_1 |_{\mathcal{E}_0^2 \to 0})$ (analytic) (3.8b)

(iii). slope of PD at limitingly small thresholds:[4] $[(dP_1/d\mathcal{E}_0^2)_{\mathcal{E}_0^2 \to 0}]_{\text{i-expt.}} = [(dP_1/d\mathcal{E}_0^2)_{\mathcal{E}_0^2 \to 0}]$ (analytic) (3.8c)

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3. It is apparent by inspection of (3.5) that using higher-order (even) moments alone always introduces one additional parameter, $\Omega_{2k}, k>2$, etc., cf. (3.10), (3.11) ff.

4. We take $\mathcal{E}_0^2$, rather than $\mathcal{E}_0$, as the variable here, because of the particular form of (3.7).

Applying Eq. (3.7) to (3.8b,c) then gives explicitly (for the limiting value $\mathcal{E}_o \to 0$):

(ii).
$$\lim_{\mathcal{E}_o \to 0} \left(\frac{1-P_1}{\mathcal{E}_o^2} \right)_{i\text{-xpt}} = \lim_{E_o \to 0} \left(\frac{1-P_1}{E_o^2} \right)_{i\text{-xpt}} \langle E_A^2 \rangle$$

$$= G_A(A_A, \Gamma_A') \equiv e^{-A_A} \sum_{m=0}^{\infty} \frac{A_A^m}{m!\,2\hat{\sigma}_{mA}^2} \tag{3.9a}$$

(iii).
$$\lim_{\mathcal{E}_o \to 0} \left(\frac{G_A + dP_1/d\mathcal{E}_o^2}{\mathcal{E}_o^2} \right)_{i\text{-xpt}} = \lim_{E_o \to 0} \left(\frac{G_A + \langle E_A^2 \rangle \cdot dP_1/dE_o^2}{E_o^2} \right)_{i\text{-xpt}} \langle E_A^2 \rangle$$

$$= H_A(A_A, \Gamma_A') \equiv e^{-A_A} \sum_{m=0}^{\infty} \frac{A_A^m}{m!\,(2\hat{\sigma}_{mA}^2)^2} = G_A^2 \tag{3.9b}$$

where the limits are non-zero (positive) [approximated in practice, of course, by taking $\mathcal{E}_o = \mathcal{E}_{o,xpt} \doteq 0$, for a $P_1 = 1-10^{-2}$, or $1-10^{-3}$, for example, cf. Sec. 5 ff.] The "approximately equal symbols ("$\cong$", "$\doteq$"), are replaced in these limits by equalities, and the parameters involved are the theoretical, or "infinite-population" parameters. Note, also that both[5] G_A, $H_A = G_A^2$ are independent of Ω_{2A}, cf. (3.8a), a further simplification resulting from the particular analytic form of the PD, (3.7). Although a search procedure is still needed, it is now greatly simplified[6] vis-à-vis the approximative

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5. The result (3.9b) is established directly by noting that for small $\mathcal{E}_o$, $P_1 = \exp(-K\mathcal{E}_o^2)$, which applied to (3.9a) gives $K = G_A$, so that $P_1 = \exp(-G_A\mathcal{E}_o^2)$. Next, using this in (3.9b) yields at once the indicated result, with the interesting identify [Eq. (3.9a)]$^2 \equiv H_A$, (3.9b).

6. In fact, from the observations noted in Appendix A-I, A.I-1, it is possible to get good "starting" values of the parameters $(A_A, \Gamma_A', \Omega_{2A})$ directly from $\langle E_A^2 \rangle$ and the PD, P_{1A}. For more refined values one can use these to initiate the computational search. Typical parametric behavior of (3.7) with Γ_A' and A_A is indicated in Figs. 3.1(II), 3.2(II), [Middleton, 1977].

"brute-force" approach using the approximate form (3.7) and any 3 points $(0<\mathcal{E}_{o1},\mathcal{E}_{o2},\mathcal{E}_{o3}<\infty)$.

We can go on now to obtain, in principle, the infinite set of parameters, Ω_{2k}, $k\geq 2$, which, with $(\Omega_{2A},A_A,\Gamma'_A)$, completely specify the exact (first-order) Class A model, whose c.f. is given by (3.1) or (3.3). Rewriting (3.5) in unnormalized form gives

$$\begin{cases} \langle E_A^2\rangle = 2\Omega_{2A}(1+\Gamma'_A) \quad , \text{ or } \Omega_{2A}(1+\Gamma'_A) = \langle E_A^2\rangle/2, \text{ and} & (3.10a) \\ \langle E_A^4\rangle = \Omega_{4A}+2[\Omega_{2A}(1+\Gamma'_A)]^2, & (3.10b) \\ \langle E_A^6\rangle = \Omega_{6A}+9\Omega_{4A}\Omega_{2A}(1+\Gamma'_A)+6\Omega_{2A}^3(1+\Gamma'_A)^3 \quad , & (3.10c) \\ \quad \vdots \qquad\qquad \vdots & \\ \langle E_A^{2k}\rangle = \Omega_{2k,A}+\Omega_{2k-2}f_2(\Omega_{2A}(1+\Gamma'_A))+\Omega_{2k-4}f_4(\Omega_{2A}(1+\Gamma'_A))+\dots & \\ \qquad \dots(\ )\Omega_{2A}^k(1+\Gamma'_A)^k \quad , & (3.10d) \end{cases}$$

from which it is immediately apparent that by computing $\langle E_A^{2k}\rangle_{i\text{-expt}}(\equiv\langle E_A^{2k}\rangle)$ from the (ideal) experimental (infinite population) data, and using (3.10a) for $\Omega_{2A}(1+\Gamma'_A)$, we can obtain all the (scale) parameters $\Omega_{2k,A}$ by iteration. For example, we have from (3.10a-c),

$$\begin{cases} \Omega_{4A} = \langle E_A^4\rangle - \frac{1}{2}\langle E_A^2\rangle^2 = \langle E_A^4\rangle - \langle E_A^2\rangle^2 + \frac{1}{2}\langle E_A^2\rangle^2 = \text{var } E_A^2 + \frac{1}{2}\langle E_A^2\rangle^2 \ (>0) & (3.11a) \\ \Omega_{6A} = \langle E_A^6\rangle - \frac{9}{2}\langle E_A^4\rangle\langle E_A^2\rangle + 3\langle E_A^2\rangle^3 \quad (>0), \text{ etc.} & (3.11b) \end{cases}$$

With $(\Omega_{2A},A_A,\Gamma'_A)$ obtained above according to (3.8), (3.9), Eq. (3.11) then provides the complete limiting experimental specification of all the parameters of the exact (first-order) Class A model. Incidentally, it is also evident from (3.10), (3.11) that we can obtain the $\Omega_{2k,A}$ $(k\geq 2)$ apart from having to determine the basic parameters $(\Omega_{2A},A_A,\Gamma'_A)$.]

3.2 Approximate Parameter Determination (Class A):

In those cases where the approximate forms (3.6), (3.7) yield acceptably close agreement with the experimental PD (and we do not wish to determine the higher order parameters $\Omega_{2k,A}$ associated with the exact model [cf. (3.1), (3.3)]), we can obtain approximate values of the 3 basic parameters, $(\Omega_{2A}, A_A, \Gamma'_A)$, from the first three even moments of the data, on replacing $\Omega_{2k,A}$ therein, cf. (3.5), (3.10), by an appropriate function of Ω_{2A}.

We first use the summed form of (3.6), viz. (3.3) with the $\ell\geq 2$ terms omitted, expand the exponential and compare with (3.1b). The result is the replacement form of $\Omega_{2k,A}$, [cf. Appendix A-I, Sec. A.1-1], e.g.

$$\Omega_{2k,A} \rightarrow A_A k! \left(\frac{\Omega_{2A}}{A_A}\right)^k \quad , \; k \geq 2 \; , \tag{3.12}$$

e.g. $\Omega_{4A} \rightarrow 2\Omega_{2A}^2/A_A$; $\Omega_{6A} \rightarrow 6\Omega_{2A}^3/A_A^2$, etc. Inserting these in (3.10a-c) then gives us

$$\begin{cases} \langle E_A^2 \rangle = 2\Omega_{2A}(1+\Gamma'_A) \quad , & (3.13a) \\ \langle E_A^4 \rangle \doteq 8\Omega_{2A}^2\left(\frac{1}{A_A} + (1+\Gamma'_A)^2\right) \quad , & (3.13b) \\ \langle E_A^6 \rangle \doteq 48\Omega_{2A}^3\left(\frac{1}{A_A^2} + \frac{3}{A_A}(1+\Gamma'_A)+(1+\Gamma'_A)^3\right) \; , \text{ etc.} & (3.13c) \end{cases}$$

These three relations can now be solved explicitly for $(\Omega_{2A}, A_A, \Gamma'_A)$. Letting $z \equiv 1+\Gamma'_A$ in (3.13) we first obtain $z = \langle E_A^2 \rangle/2\Omega_{2A} \equiv 1+\Gamma'_A$ and apply this to (3.13b,c), solving for A_A and finally Ω_{2A}. The results are easily shown to be

$$\begin{cases} \Omega_{2A} \doteq \dfrac{3(e_4-2e_2^2)^2}{4(e_6+12e_2^3-9e_2e_4)} \quad ; \quad e_{2k} \equiv \langle E_A^{2k} \rangle & (3.14a) \\ A_A \doteq \dfrac{9(e_4-2e_2^2)^3}{2(e_6+12e_2^3-9e_2e_4)} \quad ; & (3.14b) \end{cases}$$

$$\Gamma_A' \cong \frac{2e_2(e_6+12e_2^3-9e_2e_4)}{3(e_4-2e_2^2)^2} - 1 \; ; \qquad (3.14c)$$

where, of course, all these approximations must be positive to be accepted.

The advantage of these approximate relations for the three basic Class A parameters is that they do not require the search procedure involved in the "exact" calculations (3.8), (3.9). On the other hand, the disadvantages are two-fold, that: (1), the results are approximate; (2), they necessarily [cf. (3.12)] do not provide all the (first-order) parameters of the exact model (including $\Omega_{2k,A}$). [See Appendix A-I for further discussion].

4. Formal Determination of Class B Model Parameters:

From the general results of Section 3 preceeding we are now in a position to extend the analysis to the more complex situation of determining the basic parameters for Class B interference models. The situation is more complex because now the exact characteristic function, and hence the corresponding pdf and PD, w_{1-B}, P_{1-B}, must be approximated by a pair of characteristic functions, pdf's and PD's [cf. (2.4),(2.5), and the discussion in Section 2 above, generally].

The exact c.f. for Class B interference, along with an independent gaussian component, may as in the Class A cases, be expressed by equivalent series forms, in this case three, each of which is particularly suited to one of the three tasks: (i), exact determination of the (even) moments; (ii), approximation for small and intermediate envelope values, and (iii), a suitable approximation for large envelope magnitudes. From these forms we can then obtain all the desired parameters of the (first-order) Class B statistics, as we shall show below. From Eqs. (2.51), [Middleton, 1976], (2.23), [Middleton, 1977] the exact closed-form expression for the c.f. here is

$$\hat{F}_1(ia\lambda)_{B+G} = \exp\{-\sigma_G^2 a_B^2 \lambda^2/2 + A_B \int_0^\infty \langle [J_0(a_B\lambda\hat{B}_{0B})-1]\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}}, dz\} \; , \qquad (4.1)$$

with now, cf. (2b),

$$a_B^2 \equiv \{2\Omega_{2B}(1+\Gamma_B')\}^{-1} \quad ; \quad \Omega_{2B} = A_B\langle \hat{B}_{oB}^2 \rangle_{\underset{\sim}{\theta}'}/2 \quad ; \quad \Gamma_B' \equiv \sigma_G^2/\Omega_{2B}, \tag{4.1a}$$

where $\langle\,\rangle_{\underset{\sim}{\theta}'} \equiv \int_0^\infty (\;)_{\underset{\sim}{\theta}'} dz$, $d\underset{\sim}{\theta}' = A_{oB}, e_{o\gamma}', \mathcal{Q}_{RI}$, etc., i.e., all relevant, random parameters of the typical input signal envelope, as discussed in detail in Section 2 [Middleton, 1977, 1976], e.g. cf. (2.87d), (2.87c), [Middleton, 1976]. It is, of course, the analytically infinite domain of z, i.e.,($0 \leq z \leq \infty$), which reflects the physically, infinitely long-decaying transient effects, which in turn qualitatively and quantitatively distinguish Class B interference from the Class A types, cf. (Sec. (2.4), (2.5) in [Middleton, 1977].

The various series forms of (4.1) are obtained first by observing that the desired c.f. can be expressed as the limit

$$\hat{F}_1(ia\lambda)_{B+G} = \lim_{\hat{z}_o \to \infty} \hat{F}_1(ia\lambda|\hat{z}_o)_{B+G} , \tag{4.2}$$

with $\hat{F}_1(ia\lambda|\hat{z}_o)_{B+G}$ given by (4.1) with the improper integral, $\int_0^\infty(\;)dz$, replaced by the proper one, $\int_0^{\hat{z}_o}\langle J_o - 1\rangle dz$. [As noted earlier (Sec. 5.2B, [Middleton, 1976]), this permits the series expansion of the integrand.] Thus, for the first series form we have here the analogue of (3.1b):

(I):
$$\hat{F}_1(ia\lambda)_{B+G} = \exp\{-\Omega_{2B}(1+\Gamma_B')a_B^2\lambda^2/2 + \sum_{k=2}^{\infty} \frac{(a_B\lambda)^{2k}(-1)^k}{2^k(k!)^2}\Omega_{2k,B}\} , \tag{4.3}$$

where now

$$\Omega_{2k,B} = \lim_{\hat{z}_o \to \infty} A_B 2^{-k} \left\langle \int_0^{\hat{z}_o} B_{oB}(z,\lambda,\theta')^{2k} dz \right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'} = A_B 2^{-k} \int_0^\infty \langle \hat{B}_{oB}^{2k} \rangle_{\underset{\sim}{\theta}',\underset{\sim}{\lambda}} dz \tag{4.3a}$$

$$\equiv A_B \langle \hat{B}_{oB}^{2k} \rangle / 2^k , \tag{4.3b}$$

cf. (3.2a). As before, A_B is the Impulsive Index, Ω_{2B} the intensity of the nongaussian component, and Γ_B' the ratio of the intensities of the (independent) gauss to this nongaussian term.

The second, equivalent series from of the c.f. is found by the methods described in Sections (2.3)-(2.6), [Middleton, 1977], and Sections (2.3), (2.5), (2.7), [Middleton, 1976], applied to $F_1(ia\lambda|\hat{z}_0)_{B+G}$ and followed by the limit $\hat{z}_0\to\infty$, cf. (4.2). The result is Eq. (2.87), [Middleton, 1976]; Eq. (2.38), [Middleton, 1977]:

(II):
$$\hat{F}_1(ia\lambda)_{B+G} = \exp\{-b_{1\alpha}A_B(a_B\lambda)^{\alpha}-(\sigma_G^2+b_{2\alpha}A_B)a_B^2\lambda^2/2 - \sum_{\ell=1}^{\infty}(-1)^{\ell}b_{(2\ell+2)\alpha}A_B(a_B\lambda)^{2\ell+2}\}, \tag{4.4}$$

where (cf. (2.38a-d), [Middleton, 1977])

$$b_{1\alpha} = \frac{\Gamma(1-\alpha/2)}{2^{\alpha/2-1}\Gamma(1+\alpha/2)}\left\langle\left(\frac{\hat{B}_{oB}}{\sqrt{2}}\right)^{\alpha}\right\rangle \quad ; \quad b_{2\alpha} = \left(\frac{4-\alpha}{2-\alpha}\right)\frac{\langle\hat{B}_{oB}^2\rangle}{2} = \left(\frac{4-\alpha}{2-\alpha}\right)\Omega_{2B}/A_B;$$

$$b_{(2\ell+a)\alpha} = \frac{(4\ell+4-\alpha)\Omega_{(2\ell+2),B}A_B^{-1}}{\ell!(\ell+1)!(2\ell+2-\alpha)(2\ell+2)2^{\ell+1}}. \tag{4.4a}$$

Here α is the <u>spatial density-propagation parameter</u>, {cf. (2.37), (2.24), (2.26)}, [Middleton, 1977], such that $0 < \alpha < 2$ in the above. [The details of the derivation are provided in Section (2.7), [Middleton, 1976].]

The third, equivalent series form of c.f. is obtained from (4.3) in direct comparison with (3.1b). The result is

(IIIa):
$$\hat{F}_1(ia\lambda)_{B+G} = \exp\{-A_B-\sigma_G^2a_B^2\lambda^2/2+A_Be^{-a_B^2\lambda^2\Omega_{2B}}[1+\sum_{\ell=2}^{\infty}\frac{(\Omega_{2B}/A_B)^{\ell}}{2^{\ell}(\ell!)^2}(-1)^{\ell}\hat{C}_{2\ell} \cdot(a_B\lambda)^{2\ell}], \tag{4.5}$$

with $\hat{C}_{2\ell}$ given by (3.3a) on replacing A_A by A_B, etc. therein. This expression is the equivalent of (4.3). We shall need, however, an appropriate alternative to the steepest-descent form of (4.4). This is readily accomplished by the procedures described in (2.91) - (2.93), [Middleton, 1976] or (2.41)-(2.42), [Middleton, 1977], applied to (4.4), to yield

(IIIb): $$\hat{F}_1(ia\lambda)_{B+G} = \exp\{-A_B + A_B e^{-b_{2\alpha} a_B^2 \lambda^2/2} [1+\sum_{k=2}^{\infty} B_k (a_B\lambda)^{2k}] - \sigma_G^2 a_B^2 \lambda^2/2\}, \quad (4.6)$$

where the coefficients B_k are found directly by a series of recursion formulas, cf. (2.41b), [Middleton, 1977], in terms of $b_{(2\ell+2)\alpha}$.

As we have noted above, all these forms (I-IIIb) of the characteristic function are equivalent, provided, of course, that all terms in the respective series are retained. It is when we seek manageable analytic approximations to the c.f., that is, when we retain only those few terms which principally influence the structure of the associated pdf (and PD), that these different forms lose their equivalence. Because of the presence of the fractional power, λ^α, in II, (4.4), it may seem that this form of the c.f. is basically different from the others. However, a little reflection shows that this effect is counterbalanced in the (totality) of the series $\sum_\ell$, where each term is a function of α, cf. (4.4a), leaving only an (infinite) series of (even) integral powers of λ. In fact, by reversing the steps: (2.59), (2.67b) in (2.65), (2.79)-(2.87), [Middleton, 1976], we return to the generic form (4.1) from which (4.3), (4.5), are alternatively derived.

Now, before going on to determine the parameters ($\Omega_{2B}, A_B, \Gamma_B', b_{1\alpha}, \alpha, \ldots$ etc.) of the Class B model, we can use form I, (4.3), to get directly exact expressions for the even moments of the (normalized) envelope, $\mathcal{E}$, by differentiation of this c.f. with the help of (3.4). Since the form of (4.3) is identical with (3.1b) for the Class A cases, we can write at once from (3.5)

$$\langle \mathcal{E}_B^0 \rangle = 1 \ ; \ \langle \mathcal{E}_B^2 \rangle = 1; \langle \mathcal{E}_B^4 \rangle = \frac{\Omega_{4B}}{\Omega_{2B}^2 (1+\Gamma_B')^2} + 2$$

$$\langle \mathcal{E}_B^6 \rangle = \frac{\Omega_{6B}}{\Omega_{2B}^2 (1+\Gamma_B')^3} + \frac{9\Omega_{4B}}{\Omega_{2B}^2 (1+\Gamma_B')^2} + 6, \text{ etc.,} \quad (4.7)$$

with $\mathcal{E}_B = a_B E$, cf. (4.1a). [Although (4.4) is equivalent to (4.3), it is clearly inconvenient for the calculation of moments by differentiation, cf. (3.4), because of the term $(\sim\lambda^\alpha)$, which can cause divergencies (as $\lambda \to 0$) unless balanced by the totality of the series, $\sum_\ell$, cf. remarks above.]

Finally, it is convenient at this point to do a further taxonomy with regard to the model parameters, generally. We have already made a classification between global and generic parameters in [Middleton, 1976], Sec. 2.5. Here we are concerned with Class A and B global parameters which we shall further distinguish as being structure parameters, i.e., those governing the form of the PD (and pdf) primarily, e.g., (A_A, Γ_A'), $(A_B, \Gamma_B', \alpha, b_{1\alpha}, b_{2k,\alpha}, k \geq 2)$, and scale parameters, i.e., which primarily set the level and scale of the PD (and pdf), e.g. $(b_{2\alpha}, \Omega_{2k;A,B}, k \geq 1;\ N_I)$. Thus, in our approximate model forms we have (A_A, Γ_A') as structure parameters, with Ω_{2A} as sole scale parameter, while for Class B, $(A_B, \Gamma_B', \alpha, b_{1\alpha})$ are the structural parameters, with $(\Omega_{2B}, b_{2\alpha}, N_I)$ the scale parameters.

4.1 "Exact" Parameter Determination (Class B):

We begin by introducing first the two approximating c.f.'s, respectively appropriate for small and intermediate values of the envelope, and for large (and small) envelope values [cf. remarks following Eq. (3.7)]. We then employ (2.4), (2.5) suitably joined at the "bendover" point, $\mathcal{E}_B$, cf. Figs.2.1,2.2 (and Fig. 3.5, II), [Middleton, 1977, 1976], to establish the desired approximate, composite pdf and PD. From these (or the corresponding c.f's) we obtain the basic parameters of the Class B approximate analytical (first-order) PD, and pdf. The joining process whereby these six parameters may be (i), ideally, (ii), empirically determined, is then briefly discussed, extending the earlier treatment (Sec. 3, [Middleton, 1977, 1976]) to an "exact" procedure.

The two approximating c.f.'s are obtained from forms II, IIIb, Eqs. (4.4), (4.6), following the rationales described in Sec. 2.6.1, [Middleton, 1977], for example. These are respectively

$(0 \leq \mathcal{E} \leq \mathcal{E}_B)$: Eqs. (2.5a, 3.4a); [[1], Middleton, 1977]

$$\hat{F}_1(ia\lambda)_{B+G} \doteq \hat{F}(ia\lambda)_{(B+G)-I} = e^{-b_{1\alpha}A_B(a_B\lambda)^{\alpha} - \Delta\sigma_G^2 a_B^2 \lambda^2/2} \qquad (4.8)$$

$$\Delta\sigma_G^2 \equiv \sigma_G^2 + b_{2\alpha}A_B;\quad 0 < \alpha < 2; \qquad (4.8a)$$

and

$(\mathcal{E}_B \le \mathcal{E} \le \infty)$: Eq. (2.5b),(3.4b): [[1], Middleton, 1977]

$$\hat{F}_1(ia\lambda)_{B+G} \doteq \hat{F}_1(ia\lambda)_{(B+G)-II} = \exp\{-A_B + A_B e^{-b_{2\alpha} a_B^2 \lambda^2/2} - \sigma_G^2 a_B^2 \lambda^2/2\}. \tag{4.9}$$

The corresponding PD's are:

$(0 \le \mathcal{E}_0 \le \mathcal{E}_B)$: Eqs. (2.7a), (2.5,6): [[1], Middleton, 1977]

$$\hat{P}_1(\hat{\mathcal{E}} > \hat{\mathcal{E}}_0)_{B-I} \simeq 1 - \hat{\mathcal{E}}_0^2 \sum_{n=0}^{\infty} \frac{(-1)^n \hat{A}_\alpha^n \Gamma(1+\frac{\alpha n}{2})}{n!} \, {}_1F_1(1+\frac{\alpha n}{2}; 2; -\hat{\mathcal{E}}_0^2) , \tag{4.10}$$

with

$$\hat{\mathcal{E}}_0 \equiv \mathcal{E}_0 N_I / 2G_B \; ; \; G_B^2 \equiv (\frac{4-\alpha}{2-\alpha} + \Gamma_B')/4(1+\Gamma_B'); \; \hat{A}_\alpha \equiv A_B(b_{1\alpha} a_B / G_B)^\alpha , \tag{4.10a}$$

and ${}_1F_1$ a confluent hypergeometric function, and[7]

$(\mathcal{E}_B \le \mathcal{E}_0 < \infty)$: Eqs. (2.7b), (3.8): [[1], Middleton, 1977

$$P_1(\mathcal{E} > \mathcal{E}_0)_{B-II} \simeq e^{-A_B} \sum_{m=0}^{\infty} \frac{A_B^m}{m!} e^{-\mathcal{E}_0^2 / 2\hat{\sigma}_{mB}^2} , \text{ with} \tag{4.11a}$$

$$2\hat{\sigma}_{mB}^2 \equiv (m/\hat{A}_B + \Gamma_B')/(1+\Gamma_B'); \quad \hat{A}_B \equiv (\frac{2-\alpha}{4-\alpha}) A_B . \tag{4.11b}$$

[The associated pdf's, $w_{1-I,II}$, for the two regions I: $(0 \le \mathcal{E} \le \mathcal{E}_B)$, II: $(\mathcal{E}_B \le \mathcal{E} \le \infty)$, cf. Fig. 2.1, are given explicitly by Eqs. (2.8a,b) [Middleton [1],

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7. The relation (4.11a) appears without the factor $(4G_B^2)^{-1}$ included in (2.7b), (3.8), [[1], Middleton, 1977], which factor is the result of an earlier, approximate procedure [cf. last paragraph, Section 2.6.1 [[1], Middleton, 1977]] and discussion here, e.g. Sec. 4.1, (4.21) and [[4], Middleton, 1978].

1977], and (4.3), (4.4), [Middleton, 1976]. These PD's and pdf's are also proper PD's and pdf's.[8]]

The exact c.f.'s (4.3), (4.5) indicate that the basic parameters of the Class B model are: $(\Omega_{2B}, A_B, \Gamma_B', \Omega_{2k;B}(k>2))$. The equivalent (exact) c.f.'s (4.4), (4.6), provide an alternative set: $(\Omega_{2B}, A_B, \Gamma_B', \alpha, b_{1\alpha}, b_{(2\ell+2)\alpha}, \ell \geq 1)$. The approximate c.f.'s, (4.8), (4.9), which we use to obtain the approximating PD's (and pdf's), (4.10), (4.11) above require only the six parameters: $(\Omega_{2B}, A_B, \Gamma_B', \alpha, b_{1\alpha}, N_I)$.[9] Because of this, we shall determine the parameters of the c.f.'s (4.4), (4.6), here. It is now important to note that there is a variety of equivalent procedures for determining the desired six parameters of the approximating (and exact) c.f.'s. (This fact was not indicated in the earlier analyses [Middleton, 1977, 1976].) All procedures demand continuity of $P_{1-I,II}$ at $\mathcal{E}_o = \mathcal{E}_B$, i.e. condition (i), (2.1). Moreover, we may also require continuity and smoothness of the pdf's at $\mathcal{E} = \mathcal{E}_B$, according to conditions (ii), (iii), (2.1). In addition, we may require that the PD's, $P_{1-I,II}$, and their derivatives, cf. (2.1), be exact[10] at $\mathcal{E}_o = \mathcal{E}_B$, even though they may not be for other values of $\mathcal{E}_o$.

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8. A proper pdf, $w_1(\mathcal{E})$, is such that (here for envelopes $w_1 \geq 0$, $0 \leq \mathcal{E} \leq \infty$. and $\int_0^\infty w_1 d\mathcal{E} = 1$, which is equivalent to $P_1(\mathcal{E} \geq 0) = 1$, $P_1(\mathcal{E} \geq \infty) = 0$, and $P_1(\mathcal{E} > \mathcal{E}_o)$, $0 \leq \mathcal{E}_o \leq \infty$ is nondecreasing. Corresponding conditions on the c.f. are $F_1(0) = 1$, $|F_1(ia\lambda)| \leq 1$, $F_1(\infty) = 0$; this last if there are no "mass-points", or delta functions, in the pdf. Inspection of (4.3), (4.6) demonstrates that these are indeed proper c.f.'s, and hence the pdf's and PD's are likewise proper, in view of the unique (here, Hankel transform) relationship between c.f. and pdf (and PD).

9. The parameter N_I is a scaling parameter on $\mathcal{E}$, cf. (4.10a), which is required because $F_{1-(B+G)-I}$, (4.8) does not yield a finite second moment on $(0,\infty)$, and is needed to assist the joining process for $P_{1-I,II}$ both at $\mathcal{E}_o = \mathcal{E}_B$ and as $\mathcal{E}_o \to 0$, cf. Sec. (3.2.1), [Middleton, [1] 1977].

10. Although the explicit structures of the c.f.'s (4.1), (4.3)-(4.6), <u>and</u> their approximations (4.8), (4.9), insure continuity and smoothness of $w_{1-I,II}$ in $(0 \leq \mathcal{E} \leq \infty)$, this does not necessarily mean that $w_{1-I,II}$ are equal to $w_{1-exact}$ at $\mathcal{E} = \mathcal{E}_B$, without imposing this further condition.

Various reasonable rationales for combining the approximate PD's, etc. and determining the associated (six) parameters can be given. We shall consider two here, for the so-called "exact" parameter determination. First, we observe that the type-II approximation (for the c.f., P.D., (4.9), (4.11), etc.) is only exact as $\lambda \to 0$ (or $\mathcal{E},\mathcal{E}_0 \to \infty$). It is not exact here for $\lambda \to \infty$ (or $\mathcal{E},\mathcal{E}_0 \to 0$), although it is a Class A form (which is exact for actual Class A noise, cf. Sec. (3.1) preceeding), because here we have Class B interference, which even for no independent gaussian component ($\sigma_G^2=0$) does not have "gaps-in-time" and hence has no "spikes" in the pdf for $\mathcal{E}_0=0$. This means that we cannot follow a procedure similar to (3.8) above, including now the second derivative, and obtain exact values of the parameter set $\mathcal{P}_4 = (\Omega_{2B},A_B,\Gamma_B',\alpha)$ which are associated with the type II forms [(4.9), (4.11)]. [We can, of course, use such a procedure, but the resulting $\mathcal{P}_4$ are then necessarily approximate.]

In any case, we must at least suitably join the PD's at $\mathcal{E}_0=\mathcal{E}_B$, where $\mathcal{E}_B$ is the data point of inflexion of $(P_1)_{-i-expt}$ (= $P_{1-exact}$), e.g. $d^2P_1/d\mathcal{E}_B^2 = 0$. This point always exists, of course, since there must always be a finite second moment associated with $(P_1)_{-i-expt}$ $(=(P_1)_{exact})$. Thus, two relations of the needed six here are:

$$P_{1-I} = P_{1-II} = (P_1)_{i-expt.} \tag{4.12a}$$

The second relation in (4.12a) is required, in order to relate the analytic forms to the (ideal) real world. Furthermore, for a third relation, we may reasonably require that the pdf's are at least continuous at $\mathcal{E}_0=\mathcal{E}_B$, e.g.

$$\frac{dP_{1-I}}{d\mathcal{E}_B} = \frac{dP_{1-II}}{d\mathcal{E}_B}, \tag{4.12b}$$

(which is not necessarily equal to $(dP_1/d\mathcal{E}_B)_{i-expt}$ unless so specified, nor does (4.12b) follow automatically from (4.12a), in as much as $P_{1-I,II}$ are approximations, not necessarily equal at $\mathcal{E}_B \pm \epsilon$, any ϵ). Next, we may reasonably require that $P_{1-I,II}$ become equal as $\mathcal{E}_0 \to 0$. This automatically insures that their first derivatives are also equal (as $\mathcal{E}_0 \to 0$). For, let us write from (4.10), (4.11a)

$$P_{1-I} = 1-\varepsilon_o^2\Sigma_I^{(2)} + \frac{\varepsilon_o^4}{2}\Sigma_I^{(4)}+\ldots = 1-\varepsilon_o^2\frac{dP_{1-I}}{d(\varepsilon_o^2)} + \frac{(\varepsilon_o^2)^2}{2!}\frac{d^2P_{1-I}}{d(\varepsilon_o^2)^2} + \ldots;$$

$$P_{1-II} = 1-\varepsilon_o^2\Sigma_{II}^{(2)} + \frac{\varepsilon_o^4}{2}\Sigma_{II}^{(4)}+\ldots = 1-\varepsilon_o^2\frac{dP_{1-II}}{d(\varepsilon_o^2)} + \frac{(\varepsilon_o^2)^2}{2!}\frac{d^2P_{1-II}}{d(\varepsilon_o^2)^2} + \ldots.$$

Thus, we have

$$\left\{\left(\frac{dP_{1-I}}{d(\varepsilon_o^2)} \equiv\right) \Sigma_I^{(2)} = \Sigma_{II}^{(2)} \left(\equiv \frac{dP_{1-II}}{d(\varepsilon_o^2)}\right)\right\} \text{as } \varepsilon_o \to 0 , \tag{4.13a}$$

if we require $\lim_{\varepsilon_o\to 0}(P_{1-I} = P_{1-II})$. It is immediately evident that this does <u>not</u> insure that $d^2P_{1-I}/d(\varepsilon_o^2)^2$ $(\equiv\Sigma_I^{(4)}) = d^2P_{1-II}/d(\varepsilon_o^2)^2$ $(\equiv\Sigma_{II}^{(4)})$. Relating the analytic forms again to the (ideal) real world data, we thus have the additional pair of conditions

$$\lim_{\varepsilon_o\to 0}: \begin{cases} P_{1-I} = P_{1-II} = (P_1)_{i\text{-expt}} \text{, or} \\ \Sigma_I^{(2)} = \Sigma_{II}^{(2)} = \dfrac{1-(P_1)_{i\text{-expt.}}}{\varepsilon_o^2} \end{cases} \tag{4.13b}$$

The remaining condition is, naturally

$$\langle E_B^2\rangle = 2\Omega_{2B}(1+\Gamma_B') = \langle E_B^2\rangle_{i\text{-expt}}. \tag{4.14}$$

Accordingly, we may summarize our first, approximate procedure for joining and parameter approximation by:

$$\text{I.} \quad \left\{\begin{array}{l} \left.\begin{array}{l} P_{1-I} = P_{1-II} = (P_1)_{i\text{-expt}} \\ \dfrac{dP_{1-I}}{d\varepsilon_B} = \dfrac{dP_{1-II}}{d\varepsilon_B} \end{array}\right\} \varepsilon_o = \varepsilon_B \\ \lim \varepsilon_o \to 0: \; P_{1-I} = P_{1-II} = (P_1)_{i\text{-expt.}} \\ \langle E_B^2\rangle = 2\Omega_{2B}(1+\Gamma_B') = \langle E_B^2\rangle_{i\text{-expt.}} \end{array}\right\} \tag{4.15}$$

where by (4.13a) the slopes of $P_{1-I,II}$ are also automatically equal (as $\mathcal{E}_o \to 0$).

The conditions (4.15) translate specifically, with the help of (4.10), (4.11), and the use of the asymptotic form of the hypergeometric function, ${}_1F_1$, cf. (3.7), [Middleton, 1977], respectively into

$$\hat{A}_\alpha \frac{\Gamma(1+\alpha/2)}{\Gamma(1-\alpha/2)} \hat{\mathcal{E}}_B^{-\alpha}[1+0(\hat{\mathcal{E}}_B^{-2},\alpha)] = e^{-A_B} \sum_{m=0}^{\infty} \frac{A_B^m}{m!} e^{-\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2} = (P_1)_{i\text{-expt.}}, \tag{4.15a}$$

$$\frac{\hat{A}_\alpha \alpha\Gamma(1+\alpha/2)}{\Gamma(1-\alpha/2)} \left(\frac{N_I}{2G_B}\right)^{-\alpha} \mathcal{E}_B^{-\alpha-1}[1+\ldots] = \mathcal{E}_B e^{-A_B} \sum_{m=0}^{\infty} \frac{A_B^m e^{-\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2}}{m!\,\hat{\sigma}_{mB}^2} \tag{4.15b}$$

$$\left(\Sigma_I^{(2)} \equiv\right) \sum_{n=0}^{\infty} \frac{(-1)^n \hat{A}^n \Gamma(1+\frac{\alpha n}{2})}{n!} = e^{-A_B} \sum_{m=0}^{\infty} \frac{A_B^m}{m!\,2\hat{\sigma}_{mB}^2} \quad (\equiv G_B(A_B,\Gamma'_B,\alpha),\ \text{cf. (3.9a)}) \tag{4.15c}$$

$$= \lim_{E_o\to 0} \left(\frac{1-P_1}{E_o^2}\right)_{i\text{-expt}} \langle E_B^2\rangle$$

$$\langle E_B^2\rangle = 2\Omega_{2B}(1+\Gamma'_B) : \langle E_B^2\rangle_{i\text{-expt.}} \tag{4.15d}$$

Other possibilities exist: we may require all the conditions (2.1) to hold at $\mathcal{E}_o = \mathcal{E}_B$, viz. $(d^2P_{1-I}/d\mathcal{E}_B^2) = (d^2P_{1-II}/d\mathcal{E}_B^2)$, which permits us to drop the second moment relation (4.14) in (4.15), for example. Specifically, this becomes

$$\hat{A}_\alpha \frac{\alpha(\alpha+1)\Gamma(1+\alpha/2)}{\Gamma(1-\alpha/2)} \left(\frac{N_I}{2G_B}\right)^{-\alpha} \mathcal{E}_B^{-\alpha-2}[1+\ldots] = e^{-A_B} \sum_{m=0}^{\infty} (\mathcal{E}_B^2/\hat{\sigma}_{mB}^2 - 1) \frac{A_B^m e^{-\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2}}{m!\,\hat{\sigma}_{mB}^2} . \tag{4.16}$$

Other variants might set the various derivatives at $\mathcal{E}, \mathcal{E}_o = \mathcal{E}_B$ equal to their respective (exact) experimental data results, e.g. $(dP_1/d\mathcal{E}_B)_{i\text{-expt.}}$, etc.

In the above set (I) of conditions, and their variations, we have in each instance joined $P_{1-I,II}$ (and automatically their slopes) as $\mathcal{E}_o \to 0$. But since P_{1-II} is not an exact form as $\mathcal{E}_o \to 0$ nor does it per se insure the no "gaps-in-time" effect (when $\sigma_G=0$) characteristic of Class B noise, and P_{1-I}, here, we may somewhat more logically choose in place of (4.13b), the alternative conditions

$$\lim_{\varepsilon_o \to 0} : \quad P_{1-I} = (P_1)_{i\text{-expt.}} \quad ; \quad \frac{dP_{1-I}}{d\varepsilon_o^2} = \left(\frac{dP_1}{d\varepsilon_o^2}\right)_{i\text{-expt.}}, \tag{4.17}$$

so that a second alternative class of procedures becomes

$$\text{II.} \quad \left\{ \begin{array}{ll} \left. \begin{array}{l} P_{1-I} = P_{1-II} = (P_1)_{i\text{-expt.}} \\ \dfrac{dP_{1-I}}{d\varepsilon_B} = \dfrac{dP_{1-II}}{\varepsilon_B} \end{array} \right\} & \varepsilon_o = \varepsilon_B \\ \left. \begin{array}{l} P_{1-I} = (P_1)_{1\text{-expt.}} \\ \dfrac{dP_{1-I}}{d\varepsilon_o^2} = \left(\dfrac{dP_1}{d\varepsilon_o^2}\right)_{i\text{-expt}} \end{array} \right\} & \varepsilon_o \to 0 \\ \langle E_B^2 \rangle = 2\Omega_{2B}(1+\Gamma_B') = \langle E_B^2 \rangle_{i\text{-expt.}} & \end{array} \right\} \tag{4.18}$$

The explicit forms of (4.18) are available at once from (4.15a)-(4.15d) by inspection, where now specifically the derivative condition as $\varepsilon_o \to 0$ becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(2+\frac{\alpha n}{2})}{n!} \hat{A}_\alpha^n = \lim_{E_o \to 0} \left\{ \frac{1-P_1-(dP_1/dE_o^2)\langle E_B^2 \rangle}{E_o^2} \right\}_{i\text{-expt}} \langle E_B^2 \rangle \quad . \tag{4.18a}$$

Variants of II, (4.18) are similarly generated: we may drop the second moment relation and use a second derivative at $\varepsilon_o = \varepsilon_B$, cf. (4.16).

Finally, note that from the $\mathcal{P}_6$ parameters so obtained above, and in particular from the second moment relation (4.14), and with the help of the unnormalized form of (4.7), cf. (3.10), since here $\langle E_B^{2k}\rangle_{i\text{-expt.}} = \langle E_B^{2k}\rangle$, we may obtain $\Omega_{2\ell+2,B}$, cf. (4.3a,b) in a straight-forward fashion, as in the Class A cases, cf. (3.11a,b). It follows at once from this and the results of (4.15) or (4.18) in (4.4a), that the remaining model B parameters, e.g., $b_{(2\ell+2)\alpha}$, $\ell\leq 1$, are now specified, albeit now approximately, in terms of the experimentally observed $\langle E_B^{2k}\rangle_{i\text{-expt.}}$. For example, we have explicitly

$$b_{4\alpha} = \left(\frac{8-\alpha}{4-\alpha}\right)\frac{\Omega_{4B}}{2^5 A_B} = \left(\frac{8-\alpha}{4-\alpha}\right)2^{-5}A_B^{-1}\{\langle E_B^4\rangle - \tfrac{1}{2}\langle E_B^2\rangle^2\}, \tag{4.19a}$$

$$b_{6\alpha} = \left(\frac{12-\alpha}{6-\alpha}\right)\frac{\Omega_{6B}}{(3!)^2 2^4 A_B} = \left(\frac{12-\alpha}{6-\alpha}\right)(3!)^{-2}2^{-4}A_B^{-1}\{\langle E_B^6\rangle - \tfrac{9}{2}\langle E_B^4\rangle + 3\langle E_B^2\rangle^3\}, \text{ etc.} \tag{4.19b}$$

Other possibilities are

$$\text{III.}\quad \left.\begin{aligned} &P_{1-I} = P_{1-II} = (P_1)_{i\text{-expt}} \\ &\frac{dP_{1-I}}{d\mathcal{E}_o} = \frac{dP_{1-II}}{d\mathcal{E}_o} = \left(\frac{dP_1}{d\mathcal{E}_o}\right)_{i\text{-expt}} \end{aligned}\right\} @\mathcal{E}_o=\mathcal{E}_B;\quad \begin{aligned} &P_{1-I} = P_{1-II} \text{ as } \mathcal{E}_o \to 0; \\ &\langle \mathcal{E}_B^2\rangle_{1\text{-expt}} = 2\Omega_{2B}(1+\Gamma_B'). \end{aligned} \tag{4.20}$$

This insures exact values of $P_{1-I,II}$ and $w_{1-I,II}$ at $\mathcal{E}$, $\mathcal{E}_o = \mathcal{E}_B$, but not necessarily "smoothness" of $w_{1-I,II}$, and has the possible advantage of not requiring the experimental calculation of $[d^2P_1/d\mathcal{E}_o^2]\mathcal{E}_B$.

Another set of approximate relations, essentially used in the author's earlier work [Middleton, 1977, 1976, 1978], is

$$\text{IV.}\quad \left.\begin{aligned} P_{1-I} &= P_{1-II} \\ \frac{dP_{1-I}}{d\mathcal{E}_o} &= \frac{dP_{1-II}}{d\mathcal{E}_o} \\ \frac{dP_{1-II}}{d\mathcal{E}_o^2} &= \frac{d^2P_{1-II}}{d\mathcal{E}_o^2} \end{aligned}\right\} @\ \mathcal{E}_o = \mathcal{E}_B; \qquad \begin{aligned} &P_{1-I} = P_{1-II}\ ,\ \lim \mathcal{E}_o \to 0\ ; \\ &\langle E_B^2\rangle_{i\text{-expt}} = 2\Omega_{2B}(1+\Gamma_B'). \end{aligned} \tag{4.21}$$

with a modified form of the second moment condition

$$\begin{cases} \langle \mathcal{E}_B^2\rangle = 1 \doteq \int_0^{\hat{\mathcal{E}}_{oB}(=\mathcal{E}_B N_I/2G_B)} \hat{w}_1(\hat{\mathcal{E}})_I \hat{\mathcal{E}}^2[4G_B^2/N_I^2]d\hat{\mathcal{E}} + \int_{\mathcal{E}_B}^{\infty} w_1(\mathcal{E})_{II}\mathcal{E}^2 d\mathcal{E}, \text{ or} & (4.21a) \\ \langle \mathcal{E}_B^2\rangle = 1 \doteq N_{II} \int_0^{\infty} \mathcal{E}^2 w_1(\mathcal{E})_{II} d\mathcal{E}\ ,\ N_{II} = (4G_B^2)^{-1}\ ,\ \text{cf. (4.10a).} & (4.21b) \end{cases}$$

to give the required six relations. Equation (4.21b) is the approximate second-moment relation actually used in [Middleton, 1977], cf. Sec. 3.2.1, and [Middleton, 1976], Sec. 3.2-A, which employs a suitably normalized pdf, $w_{1-II} \cdot N_{II} \equiv (w_{1-II})_{norm}$, to determine $\mathcal{E}_B^2$ over the entire range $(0 \leq \mathcal{E} \leq \infty)$, and which can thus produce errors of somewhat too small values of P_D for large $\mathcal{E}_o$, $\mathcal{E}$, when Γ_B' is small. The relation (4.21a), although more computationally involved, is more precise and more logically motivated.]

Finally, even when the "turning point" $\mathcal{E}_B$ is not empirically available, it is still possible to obtain all the model parameters approximately. This is accomplished as follows: choose as the required six relations for $\mathcal{P}_6$

$$\text{V.}\quad \lim_{\mathcal{E}_o \to 0} \begin{cases} P_{1-I} = P_{1-II} = (P_1)_{i\text{-expt}} \\ \dfrac{dP_{1-I}}{d\mathcal{E}_o^2} = \left(\dfrac{dP_1}{d\mathcal{E}_o^2}\right)_{i\text{-expt}} \end{cases} ; \quad \begin{aligned} &\langle E_B^2\rangle_{i\text{-expt}} = 2\Omega_{2B}(1+\Gamma_B')(=\langle E_B^2\rangle), \\ &\text{and} \\ &P_{1-I} = (P_{1-})_{i\text{-expt.}}\ @\ \mathcal{E}_{o1}, \mathcal{E}_{o2} > 0, \end{aligned} \tag{4.22}$$

this last for ε_o, ε_{o2}, say, when the curve for P_{1-I} noticeably departs from the raleigh limit (as $\varepsilon_o \to 0$). The turning point, ε_B, may be approximately deduced by solving ($P_{1-I} = P_{1-II}$), $\varepsilon_o \to \varepsilon_B$ (>0), for ε_B, once the parameter set, $\mathcal{P}_6$, has been obtained according to V,(4.22).

In all these cases, of course, there is some choice, which in practical applications we may make as a matter of convenience (as long as we explicitly indicate the particular procedures used). The fact that there is no unique selection of relations for determining the (approximate) parameters $\mathcal{P}_6$ stems from the fact that the c.f., and resulting PD's are themselves necessarily approximate. In practice, of course, we perforce deal with approximate forms and data, with a consequent further degradation of accuracy. However, at the present level of accuracies these various approaches [including those discussed above] appear to give acceptable values for the parameters, in the sense of acceptably close agreement between the data curves and the associated analytical approximations [cf. Figs. (2.1)-(2.8), Part I, of Ref. 2 here]. [See also, Appendix A-I, Sec. A.1-2, for direct "start-up" estimates of P_6'].

5. Some Procedures for Obtaining Parameter Estimates with Finite Data Samples:

In the above analysis we have assumed that we have been dealing with the idealized, limiting situation of infinite data sample populations, so that the analytic and empirical statistics (moments, pdf's, etc.) are equal (probability 1). The consequence of this, of course, is that the various parameter estimations are precise (in this sense), e.g. (3.9a,b), (3.10), etc. yield exact values. Specifically, we may write for the $2k^{th}$ moments of the envelope E

$$\langle E^{2k} \rangle = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} E_j^{2k} \equiv \langle E^{2k} \rangle_{i\text{-expt}}, \quad k \geq 1, \tag{5.1}$$

in terms of the n sample values E_j, (j=1,...,n), and so on for $\Omega_{2k,A}$, (3.11), $\Omega_{2k,B}$, $b_{2k,\alpha}$, (4.13), (4.14), etc.

Practically, however, we are always limited to the empirical situation of finite data samples ($n<\infty$). This insures that our estimates are never precise (prob. 1), albeit that they may be good approximations. Thus, we have now the sample moments

$$\langle E^{2k} \rangle_e \equiv \frac{1}{n} \sum_{j=1}^{\infty} E_j^{2k} \quad , \quad k \geq 1 \; , \tag{5.2}$$

for the empirical estimates of the moments $\langle E^{2k} \rangle$. Similarly, although we may use the exact analytic forms for P_{1-A}, $P_{1-I,II}$, and their derivatives etc., as $\mathcal{E}_0 \to 0, \mathcal{E}, \mathcal{E}_0 \to \mathcal{E}_B$ in the procedures (3.9a,b), (4.15), (4.15d), and Sec. (4.2), cf. Sec. (3.1), (4.1), (4.2), it is always the experimental values of P_1, $dP_1/d\mathcal{E}_0$, etc. which are necessarily approximate, so that the various structure parameters ($A_A, \Gamma'_A, A_B, \ldots$, etc.) are consequently approximations, as well.

Letting $\gamma(G|\underset{\sim}{E}_n) \equiv$ estimator of G based on the data set ($\underset{\sim}{E}_n = \{E_j\}$, j=1,...,n), we can use (5.2) directly in (3.11a,b) to write explicitly in terms of the (even) sample moments

$$\gamma(\Omega_{4A}|\underset{\sim}{E}_n) = \langle E_A^4 \rangle_e - \frac{1}{2} \langle E_A^2 \rangle_e^2 \quad (>0) \; , \tag{5.3a}$$

$$\gamma(\Omega_{6A}|\underset{\sim}{E}_n) = \langle E_A^6 \rangle_e - \frac{9}{2} \langle E_A^4 \rangle_e \langle E_A^2 \rangle_e + 3\langle E_A^2 \rangle_e^3 \quad (>0), \text{ etc.} \tag{5.3b}$$

for the estimators of Ω_{4A}, Ω_{6A}, etc., based on the finite data samples $\underset{\sim}{E}_n (\equiv E_1, \ldots, E_n)$, $n < \infty$. Similar results apply for $\gamma(\Omega_{2k,B}|\underset{\sim}{E}_n)$ directly using (3.10), (3.11), cf. (4.14) etc., and for $\gamma(b_{2k,\alpha}|\underset{\sim}{E}_n)$ cf. (4.19a,b). The estimators, $\gamma(A_A|\underset{\sim}{E}_n), \gamma(\Gamma'_A|\underset{\sim}{E}_n), \gamma(\alpha|\underset{\sim}{E}_n)$, etc., for the various structure parameters, however, are not explicit functions of the sample data, $\underset{\sim}{E}_n$, but are, rather, functionals of these data[11] through P_1, $P_{1-I,II}$, etc., cf. (3.9)

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11. An exception arises in the case of Class A noise when the approximation (3.12) is used for $\Omega_{2k,A}$. Then the structure parameters are given approximately but explicitly in terms of the sample moments, e.g.

$$\gamma(\Omega_{2A}, \text{or} A_A, \text{or} \Gamma'_A|\underset{\sim}{E}_n) \doteq f((e_2)_e, (e_4)_e, (e_6)_e) \; ,$$

according to (3.14a,b,c), where now $(e_{2k})_e \equiv \langle E_A^{2k} \rangle_e$ here. Questions of accuracy and robustness (cf. Sec. 5.2. ff.) remain, of course.

(4.15), (4.15d) This leads to quite different statistical procedures when we address ourselves to the critical problem of the accuracy and robustness of the parameter estimates, as we shall briefly indicate below.

5.1 Remarks on the Accuracy of the Estimates:

Before proceeding further here we must remember that we are dealing always with empirical data, which in turn is always limited in quantity. To begin with, we do not generally know whether these data belong to the same statistical population, i.e., are generated by a common statistical mechanism (stationary or not [Middleton, 1960]; and (5.14), and Sec. 5.2 below). In addition, it is often not clear that the individual elements (E_j, say) of the sample, $\underset{\sim}{E}_n$, are statistically independent, a condition which must be established if the usual, convenient and effective, statistical methods of validating and analyzing the data are to be successfully employed. Thus, we must first validate the data [Middleton, 1969], which consists essentially of the following procedures, in sequence:

(5.4)
(i). test the sample data ($\underset{\sim}{E}_n$) for statistical independence[12] (of the E_j vis-à-vis the $E_{k,j\neq k}$);
(ii). test the sample data for "homogeneity", i.e., whether or not the (E_j) belong to the same statistical population.

[One such test for (i) is the "runs test" [Middleton, 1969]; for (ii),

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12. In the data sampling process, to insure independent samples we must always be careful to remove any periodic (or dc-"steady") component. Thus, if we sample instantaneous (IF) amplitudes ($\underset{\sim}{X}_n$), in the receiver, we must use "jittered" sampling, i.e., we must sample randomly in time (with no interval between samples less than (approximately) the reciprocal of the ARI (aperture×RF×IF receiver bandwidth), cf.[Plemons, et al, 1972], or more conveniently, we may sample periodically at a period incommensurable with any periodicities in the data sample, a matter of experimental testing. If we use envelope data ($\underset{\sim}{E}_n$), on the other hand, we must remove the (sample) mean, e.g. use $e_j \equiv E_j - \langle E \rangle \doteq E_j - \langle E \rangle_e$ as the typical element of the sample. In addition, if there are periodicities, we can apply the "jittered" or incommensurable-periodic sampling approach suggested for the instantaneous amplitude above.

non-parametric tests, like the Kolmogorov-Smirnov [Middleton, 1969] are particularly useful because of their small-sample (small n) as well as large sample, capabilities. A concise description and illustrated application to a class of physical problems is given in [Plemons, et al, 1972].]

Having validated the data according to (5.4) above as a first step in estimating the accuracy of the measurement of the model parameters, we next apply a combination of the classical theory of sample statistics [Cramér, 1946], [Wilks, 1962] and "goodness of fit" tests [Middleton, 1969] to the various scale parameters, i.e. the 2k-moments $\Omega_{2k,A,B}$; and to the structure parameters ($A_A, A_B, \Gamma_A', \Gamma_B', \alpha, b_{1\alpha}, b_{2k-B}$, etc.) of the Class A and B models here.

We begin with the mth order moments of the data themselves (for example, the independent, instantaneous amplitudes, $\underset{\sim}{X}_n$, with $\langle X \rangle = 0$). In first applying standard techniques of sample statistics we may start by determining the various ensemble (or statistical) averages ($\langle\ \rangle$) of these finite samples. We readily see for these independent sample values that

$$\langle\langle X^m\rangle_e\rangle = \langle\frac{1}{n}\sum_j^n X_j^m\rangle = \frac{1}{n}\sum_j^n\langle X_j^m\rangle = \langle X^m\rangle,\ m = 2k, 2k+1, \tag{5.5}$$

(since all X_j have the same pdf's), so that all these ensemble means of the sample data moments, odd as well as even, are the same as the ensemble moments. The same result applies for the adjusted envelope values, $e_j = E_j - \langle E\rangle$, which are now also statistically independent, since the "dc" component $\langle E\rangle$ has been removed. Accordingly, we say that $\langle X^m\rangle_e$, $\langle e^m\rangle_e$, considered as estimates of the mth-order moments of (X,e) are unbiased, and independent of sample size n, as well. For envelope data, $\underset{\sim}{E}_n$, it is also evident that

$$\langle\langle E^m\rangle_e\rangle = \langle\frac{1}{n}\sum_j^n E_j^m\rangle = \langle E^m\rangle\ ,\quad m = 2k,\ 2k+1\ ,\ (\text{with } \langle E\rangle > 0), \tag{5.5a}$$

so that the moment estimates of the mth-order ensemble moment of E are likewise unbiased, and independent of sample size, as well. The variances, and higher-order moments, of these sample moments, however, do depend on sample size: it is easily shown (in the case of independent samples) that

$$\mathrm{var}\{\langle X^m\rangle_e, \langle e^m\rangle_e\} \equiv \left\{ \begin{matrix} \langle\langle X^m\rangle_e^2\rangle - \langle\langle X^m\rangle_e\rangle^2 \\ \langle\langle e^m\rangle_e^2\rangle - \langle\langle e^m\rangle_e\rangle^2 \end{matrix} \right\} = \frac{\mathrm{var}(X^m, e^m)}{n}. \tag{5.6}$$

Higher order statistics are similarly derived. The result (5.6) shows that $\lim_{n\to\infty} \mathrm{var}(X^m,e^m)\to 0$: the variance of the sample m-order moment vanishes $O(n^{-1})$, or equivalently, that the estimates $[\langle X^m\rangle_e, \langle e^m\rangle_e]$ of the moments $[\langle X^m\rangle, \langle e^m\rangle]$ improve, from the point of view of a variance measure, $O(1/\sqrt{n})$ as sample size increases - a well-known result, of course; [Cramer, 1946], [Wilks, 1962].

However, for the envelope data $(\underset{\sim}{E}_n)$, which are correlated through their d.c. (and any possible periodic component[12]), we have instead of (5.6)

$$\mathrm{var}\langle E^m\rangle_e = \frac{1}{n}\langle E^{2m}\rangle - \langle E^m\rangle^2 + \frac{1}{n^2}\sum_{k\ell}^{n}{}' \langle E_k^m E_\ell^m\rangle \;, \tag{5.7}$$

and $\langle E_k^m E_\ell^m\rangle \neq \langle E_k^m\rangle\langle E_\ell^m\rangle$. It is not now necessarily true that $\lim_{n\to\infty} \mathrm{var}\langle E^m\rangle_e \to 0$, as in (5.6). Similar remarks apply for the higher order statistics of the envelope moment estimates. This behaviour reinforces the utility of working with <u>independent</u> data samples, whenever possible, cf. (5.4).

With the above in mind we can proceed to apply the methods of standard sample statistics to obtain estimates of the accuracy of our (point) estimators of the data moments $\langle E^m\rangle$, or $\langle X^m\rangle$. For example, using the instantaneous amplitude data, $\underset{\sim}{X}$, or the adjusted envelope data, $\underset{\sim}{e}$, we may replace the ensemble variances, var $\{X^m, e^m\}$ by the sample variances, $(\mathrm{var}_e X^m, \mathrm{var}_e e^m)$ in (5.6) to get

$$\mathrm{var}\{X^m, e^m\} \doteq \mathrm{var}_e(X^m, e^m)/n \;, \quad [\text{with } \mathrm{var}_e Y \equiv \frac{1}{n}\sum_j \{Y_j^2 - (\frac{1}{n}\sum_j Y_j)^2\} \;, \tag{5.8}$$

as a measure of how close the particular data estimate $\langle X^m\rangle_e$ or $\langle e^m\rangle_e$ is to $\langle X^m\rangle$, or $\langle e^m\rangle$. This measure is, of course, subject to fluctuations when considered over subsets of the infinite data population, and is thus not invariant of the particular data set employed. Continuing along this line, however, one can further develop the standard approach, at least in principle, to derive the pdf's of these moment estimators[13] and from them obtain the more meaningful interval estimates[13] associated with these original estimates [15, Middleton, 1965]. These are the probabilities that particular (point estimates, e.g., $\gamma(Y|\underset{\sim}{X}^{(k)})$, for given particular data $X^{(k)}$, (k=1,2,...), fall within $(1\pm\lambda)100\%$ of the true value of the quantity Y being estimated. Finally, by assigning a suitable cost function to measure the error between the true and estimated values, one can determine the average cost (or risk) associated with the use of the point estimator $\gamma(Y|\underset{\sim}{X})$, considered over all possible data sets $\underset{\sim}{X} = (\underset{\sim}{X}^{(k)}, k,=1\ldots\infty))$. The interval estimation yields a probability which is a measure of the efficiency of the point estimator in any particular application (any particular $\underset{\sim}{X}^{(k)}$); the average risk measures the expected cost or average error in the use of the point estimator, over all $\underset{\sim}{X} = (\underset{\sim}{X}^{(k)}, k=1,\ldots\infty)$.

There are, however, two major technical difficulties in the direct application of standard sampling theory techniques, and its extentions, above, to the problem of estimating our noise model parameters from finite data samples. These are: (i), the nonindependent character of the direct envelope data, E_n, which are the most convenient in practice; and (ii), the analytical complexity of the estimators themselves, which are in the case of the structure parameters $(A, \Gamma', \alpha$, etc.) even more involved functions of the observed data ($\mathcal{E}$, or threshold $\mathcal{E}_0$) cf. (3.7), (4.10), (4.11). The former (i), can be largely removed by using the adjusted envelope $\{e_j = E_j - \langle E \rangle\}$, but the latter, (ii), requires an indirect alternative approach, as we shall note directly below.

Thus, for (i) let us consider the mean of the explicit estimator (5.3a) of the scale parameter Ω_{4A}:

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13. We distinguish between an estimate of Y based on a specific data set $\underset{\sim}{X}^{(k)}$, $\gamma(Y|\underset{\sim}{X}^{(k)})$, and the estimator of Y, $\gamma(Y|\underset{\sim}{X})$, based on the entire ensemble $\underset{\sim}{X}$ of data. The former is a fixed number; the latter is a random variable [Middleton, [9], p. 942.]

$$\langle \gamma(\Omega_{4A}|\underset{\sim}{E}_n)\rangle = \langle\langle E_A^4\rangle_e\rangle - \frac{1}{2}\langle\langle E_A^2\rangle_e^2\rangle$$

$$= \langle E^4\rangle - \frac{1}{2n^2}\left\langle \sum_k^n E_k^2 E_\ell^2\right\rangle \neq \langle E^4\rangle - \frac{1}{2}\langle E^2\rangle^2. \tag{5.9}$$

This estimator is generally biased and depends on sample size (n). Similar remarks apply for $\gamma(\Omega_{2k,A,B}|\underset{\sim}{E}_n)$ and all the various possible moments of these estimators. Even using $E_j = \langle E\rangle + e_j$ therein does not remove either bias or dependence on sample size, although the structure of the various correlations [$\langle E_k^2 E_\ell^2\rangle$, etc., cf. (5.9)] is made explicit. If the instantaneous amplitude data $\underset{\sim}{X}_n$ are used instead of the envelope data $\underset{\sim}{E}_n$, we can take advantage of sample independence, and the relation

$$\langle X^{2k}\rangle = \frac{(2k)!}{2^{2k}(k!)^2}\langle E^{2k}\rangle, \tag{5.10}$$

cf. {(5.11b), [Middleton, 1976]}, to obtain (for both Class A and B noise), cf. (3.10),

$$\langle X^2\rangle = \Omega_2(1+\Gamma') \; ; \; \langle X^4\rangle = \frac{3}{2}\Omega_4 + 3\Omega_2^2(1+\Gamma')^2$$

$$\langle X^6\rangle = \frac{5}{2}\Omega_6 + \frac{45}{2}\Omega_4\Omega_2(1+\Gamma') + 15\Omega_2^3(1+\Gamma')^3, \text{ etc.,} \tag{5.11}$$

which in turn gives

$$\Omega_4 = \frac{2}{3}\{\langle X^4\rangle - 3\langle X^2\rangle^2\} , \tag{5.12a}$$

$$\Omega_6 = \frac{2}{5}\{\langle X^6\rangle - 15\langle X^4\rangle\langle X^2\rangle + 60\langle X^2\rangle^3\} , \text{ etc.} \tag{5.12b}$$

the equivalent of (3.11a,b) earlier. Considering now $\gamma(\Omega_4|\underset{\sim}{X}_n)$ based on (5.12a) with $\langle\,\rangle$ replaced by the sample means $\langle\,\rangle_e$ we see at once that from (5.5) et seq.

$$\left\langle \gamma(\Omega_4|\underset{\sim}{X}_n)\right\rangle = \frac{2}{3}\left\langle\langle X^4\rangle_e\right\rangle - 2\left\langle\langle X^2\rangle_e^2\right\rangle = \frac{2}{3}\langle X^4\rangle - 2\left\{\frac{\text{var } X^2}{n} + \langle X^2\rangle^2\right\}$$

$$\neq \frac{2}{3}\langle X^4\rangle - 2\langle X^2\rangle^2 , \qquad (5.13)$$

so that even here γ is still biased, as long as sample size is finite ($n<\infty$). Higher order moments and averages are correspondingly more complex, with the estimators remaining biased ($n<\infty$).

We can, of course, for example, still calculate var $\gamma(\Omega_{2k}|\underset{\sim}{E}_n)$, cf. (5.8), substituting the sample variances (and other sample moments) for the ensemble (or "true") variances, moments, etc. to obtain measures of estimate accuracy. These suffer from bias, dependence on sample-size, complexity, and the basic fact that they are random quantities, not probabilities which measure accuracy in a consistently meaningful fashion. Finally, to develop such really indicative measures of accuracy as interval estimators and average error measures here is a formidable and as yet unachieved task. The situation is even worse for the structure parameters, because of their vastly more complex dependence on $\underset{\sim}{X}_n$, or $\underset{\sim}{E}_n$, which in turn probably precludes any directly useful analytic results. All of which leads us to the following, indirect approach:

(5.14). <u>"Goodness-of-Fit" Approach:</u> Since we know the (approximate) <u>analytic</u> form of the PD's, $P_{1-A,B}$, we use a "goodness-of-fit" procedure, whereby we test how well these analytic PD's fit the experimental PD data, when the <u>experimental</u> parameter estimates are used in the analytic forms [Middleton, 1969], [Plemons et al, 1972].

This approach is an appropriately natural one here in view of the fact that our methods above [cf. Secs. 2-4] for estimating the parameters of the approximate noise models all depend directly and explicitly on the analytic and empirical PD (or pdf), as well as on second moment calculations, a situation which is not fundamentally altered in determining all the parameters of the exact models. Again, the Kolmogorov-Smirnov tests [Middleton, 1969], [Plemons et al, 1972] are particularly useful here, especially for the small-sample conditions attending the acquired data at small probabilities (i.e., the "rare-events"), when P_1 is $O(10^{-4}$ or less). [Furthermore, when the parameter estimates are used in the K-S test, in place of the true (exact, infinite population values), it is known that such K-S tests are conservative, i.e., when the test statistic used in the test, Z_{sample}, exceeds Z_{α_o} (the appropriate threshold) the null hypothesis, H_o, (i.e., that the sample PD "fits" the analytic PD (with the estimated parameters)) may be even more safely rejected than in the strict case where the analytic PD employs the true parameters (at the same test level, i.e., the probability of falsely rejecting H_o)([Middleton, 1969], p. 38).]

5.2 Remarks on Robustness and Stability:

In all our dealings with analytic models and empirical data the notion of robustness is important. By "robustness" is meant, essentially, the following here: that small changes in parameter values produce correspondingly small changes in the underlying PD (or pdf). If large changes result, then the model is not robust, and two possibilities arise: (1), the model is basically correct, and the lack of robustness is truly characteristic; or (2), there is some important element missing (in the model) which, if included, would eliminate this spurious sensitivity, i.e., restore robustness. Accordingly, in such instances we should reëxamine the model. From the viewpoint of parameter estimation, robustness is a measure of how sensitive or insensitive the PD (or pdf) is to parameter inaccuracies. An analytic perturbation of model parameters, e.g. $\Gamma_A' \to \Gamma_A' + \Delta\Gamma'$, $A_A \to A_A + \Delta A_A$, etc. and subsequent computational evaluation is one method of detecting sensitivities and determining ranges of changes ($\Delta\Gamma'$, etc.) wherein robustness may still remain. Applying these changed values to the "goodness-of-fit" approach, (5.14), above, thus provides a probabilistic measure of robustness or its lack. Of course, there always remain the inevitable subjective element, expressed here in our choice of test threshold, (choice of probability, α). This, however, is always explicit in properly formulated probability measures involving data with random components ([Middleton, 1960], Chapter 18).

In addition to the question of robustness of the model there is that of the model's stability: does the underlying probabilistic mechanism remain invariant during the data acquisition time (and during any period for which we wish to use the model), or does the mechanism change, as reflected, for instance, in changes in the form and magnitude of the PD (or pdf), in the values of the model parameters, etc? This is a particularly pertinent question when long data acquisition periods are required, i.e., to provide enough data to establish the "tails" or "rare-event" probability portions of the PD (and pdf). The "stability" problem is essentially the same problem of establishing the validity of the data, cf. (5.4): do the acquired data belong to the same statistical population? Accordingly, suitable "tests of homogeneity" are required here to establish this fact, the (non-parametric) Kolmogorov-Smirnov test being particularly effective here, especially when one has to deal with the small-sample cases for the "rare-event" portion of the PD (and pdf); [Middleton, 1969], [Plemons et al, 1972].

6.0 Principal Results and Comments

The central purpose of this study has been to provide methods for determining the parameters of the approximate and exact analytic (first-order) models of Class A and B noise, and to indicate procedures for determining the accuracy of parameter estimates in the practical situation when only finite data samples are obtainable. The principal new results of this study may be summarized concisely:

(1). We have shown how to determine precisely the basic (first-order) parameters of the approximating Class A model, in the ideal, limiting case of infinite (homogeneous) data populations, where in principle the analytic and empirical models are equivalent [cf. Secs. 2, 3.1, 4.1]. [The basic parameters for the approximate Class A models are $\mathcal{P}'_3 = (\Omega_{2A}, A_A, \Gamma'_A)$]. However, it is possible to obtain only approximate values of the basic parameters $\mathcal{P}'_6 = (\Omega_{2B}, A_B, \Gamma'_B, \alpha, b_{1\alpha}, N_I)$ in the Class B cases, [except for Ω_{2B}, which can be precisely determined ideally].

(2). We have also shown how to determine in principle all the (first-order) parameters of the exact Class A models, and all the (even, first-order) moments of the excat Class B models, in the ideal limiting case of infinite data populations, in addition to the basic parameters noted above in (1). These are $\{\Omega_{2k,A}\}$, cf. (3.11); $\Omega_{2k,B}$, cf. (4.7); and $\{b_{2k,\alpha}\}$, (4.19), $k \geq 2$, cf. Secs. 3.1, 4.1.

(3). In the case of Class A models, we have obtained explicit approximate results for the three basic parameters $\mathcal{P}_3 = (\Omega_{2A}, A_A, \Gamma'_A)$, in terms of the first three even moments of the envelope, cf. Sec. 3.2; [Appendix A-1].

(4). We have demonstrated various alternative, approximate methods of parameter determination in Class B cases, cf. Sec. 4.1. Even if an empirical "turning-point" datum, $\mathcal{E}_B$, is not available, it is still possible to obtain all (first-order) model parameters to good approximation [Appendix A-1].

(5). We have indicated procedures for estimating parameter values in the practical situation of finite data samples [Sec. 5]. Various statistics, e.g., the variance, of sample moment estimates can be useful here. But, because of bias, dependence on sample size, and particularly the complexity of the parameter estimates' dependence on the data set $(\underset{\sim}{X}_n, \underset{\sim}{E}_n)$, it is necessary, and natural, to employ suitable "goodness-of-fit" tests. The parameter estimates (as determined above in (1) - (4)) regarded as the "true" values, are used therein to obtain probability measures of the accuracy, or more precisely, the acceptability with a given statistical control (choice of α_0, the significance level of the test), of the parameter estimates, cf. Sec. (5.1).

In dealing with the crucial, i.e. small-probability or "rare-event" region of the PD (or pdf) where much of the distinguishing nongaussian character of man-made and natural noise phenomena appear, we must usually employ small-sample statistical methods, e.g. Kolmogorov-Smirnov tests, etc., since data are often relatively limited here, in order to avoid encountering possible lack of stability in the data source itself: too long periods of observation can lead to the acquisition of inhomogeneous data [cf. (5.4), and Sec. (5.2)].

The general procedure for judging the accuracy, robustness, and stability of the parameter estimates, as outlined in Secs. 5.1, 5.2, are:

(i). First, validate the data, cf. (5.4), i.e., establish independence and homogeneity of the data sample.

(ii). Second, use suitable "goodness-of-fit" tests (Kolmogorov-Smirnov, etc.), with the sample estimates, to determine whether or not the analytic model, with these empirical estimates, acceptably (i.e. with choice of significance level, α_0)"fits" the empirical PD (or pdf).

(iii). Third, for robustness (or its lack), vary the parameters in the analytic model, e.g., $A_A \to A_A + dA_A$, etc., computationally to determine the extent of change in the PD (and pdf), with particular attention, of course, to the nongaussian regions.

(iv). Fourth, stability (data homogeneity) is established in the course of (1), validating the data. An important question here is how long is the period before instability noticeably sets in. (This will depend, of course, on the specific physical interference mechanisms involved, e.g., automobiles, mobile communications, etc., and their usage periods.)

Finally, further questions clearly remain: (1), the specific, quantitative application of the results and procedures described here to empirical data; (2), the further development of an estimation and sample statistics theory based explicitly on these Class A and B models; (3), the possibility of a tractable theory of optimum estimation (along the lines discussed in Chapter 21 [Middleton, 1969], Chapter 3 [Middleton, 1965], for example); (4) empirical examination of the robustness and stability questions, in applications, and so on. Subsequent studies in this series [Middleton, [1]-[5]] will address one or more or these problems, as the work develops.

Appendix A-I:

SOME PRELIMINARY PROCEDURES FOR INITIAL ESTIMATION OF THE PARAMETERS OF THE NOISE MODEL

Frequently (usually for Class B noise) a "mesh" or curve-fitting procedure involving parameter search and localization based on the rationales described in Eqs. 3.8, 3.9 for Class A, and Eqs. (4.15), (4.18), (4.20)-(4.22) for Class B noise, must ultimately be used in practise to establish parameter values. Nevertheless, it is possible to obtain reasonably accurate "ball-park", or "start-up", values of these parameters if we take advantage of a variety of (computationally) empirical observations, which we shall describe briefly below. These observations are fairly complete and direct for Class A interference, and not so direct or complete in the Class B cases. However, in both instances they offer useful starting procedures, which can very considerably reduce the labor of parameter estimation, particularly entailed, for example, in the various "joining" procedures required for the double approximations of Class B.

A.1 Preliminary Procedures for Parameter Estimation of Class A Noise:

A direct procedure based on the first three even moments of the envelope, according to (3.14), provides explicit, approximate estimates of $\mathcal{P}_{3A} = (A_A, \Gamma'_A, \Omega_{2A})$. The basic approximation here is (3.12), which can be shown to be equivalent to calculating the (even) moments $\langle E^{2n}\rangle_A$ using the approximate pdf:

$$w_1(E)_A = -\left.\frac{dP_1}{dE_o}\right|_{E_o=E} \doteq e^{-A_A} \sum_{m=0}^{\infty} \frac{A_A^m E\, e^{-E^2 a_A^2/2\hat{\sigma}_{mA}^2}}{m!(\sigma_{mA}^2/a_A^2)} = w_1(E)_{A\text{-approx}} \qquad a_A^2 = \{2\Omega_{2A}(1+\Gamma'_A)\}^{-1}, \tag{A.1-1}$$

derived from (3.7). The result is

$$\langle E^{2n}\rangle_A \doteq \frac{n!e^{-A_A}}{a_A^{2n}} \sum_{m=0}^{\infty} \frac{A_A^m}{m!} (2\hat{\sigma}_{mA}^2)^n , \tag{A.1-2}$$

which yields (3.13), $n \geq 1$, since the resulting series (in m) are readily found with the help of

$$\sum_{m=0}^{\infty} \frac{m^k A_A^m}{m!} = \left(A_A \frac{d}{dA_A}\right)^k \sum_{m=0}^{\infty} \frac{A_A^m}{m!} = \left(A_A \frac{d}{dA_A}\right)^k e^{A_A} . \qquad \text{(A.1-3)}$$

One useful check on the resulting numerical values in (3.13), where the $\langle E^{2k}\rangle_{xpt}$ are empirical, is first to obtain $\mathcal{P}_3 = (A_A, \Gamma'_A, \Omega_{2A})$ from (3.13), and then to compute $\langle E\rangle$, $\langle E^3\rangle$, $\langle E^5\rangle$, etc. using $w_1(E)_{A\text{-approx.}}$, (A.1-1), with these $\mathcal{P}_3$. These results, in turn, are compared with $\langle E\rangle_{xpt}$, $\langle E^3\rangle_{xpt}$, $\langle E^5\rangle_{xpt}$, etc., obtained from the empirical envelope data. Analytically we have from (A.1-1)

$$\langle E^{2n+1}\rangle \doteq e^{-A_A} \frac{(2n+1)!\sqrt{\pi}}{2^{2n+1} n!\, a_A^{2n+1}} \sum_{m=0}^{\infty} \frac{A_A^m}{m!} (2\hat{\sigma}_{mA}^2)^{n+1/2} , \quad n \geq 0, \qquad \text{(A.1-4)}$$

which becomes specifically

$$(n=0): \quad \langle E\rangle \doteq \frac{\sqrt{\pi}}{2a_A} e^{-A_A} \sum_{m=0}^{\infty} \frac{A_A^m}{m!} (2\hat{\sigma}_{mA}^2)^{1/2}, \qquad \text{(A.1-4a)}$$

$$(n=1): \quad \langle E^3\rangle \doteq \frac{3\sqrt{\pi}}{4a_A^3} e^{-A_A} \sum_{m=0}^{\infty} \frac{A_A^m}{m!} (2\hat{\sigma}_{mA}^2)^{3/2} , \qquad \text{(A.1-4b)}$$

$$(n=2): \quad \langle E^5\rangle \doteq \frac{15\sqrt{\pi}}{8a_A^5} e^{-A_A} \sum_{m=0}^{\infty} \frac{A_A^m}{m!} (2\hat{\sigma}_{mA}^2)^{5/2} , \text{ etc.} \qquad \text{(A.1-4c)}$$

One natural criterion of accuracy here is

$$\left|\langle E^{2n+1}\rangle_{approx\text{-}analyt.} - \langle E^{2n+1}\rangle_{xpt}\right| / \langle E^{2n+2}\rangle_{xpt}^{\frac{2n+1}{2n+2}} \leq B_n(\ll 1), \qquad \text{(A.1-5)}$$

when we may say that values of $\mathcal{P}_3$ determined from (3.13) are "good" approximations to the true values as long as B_n is "small" in some sense, usually vis-à-vis unity, cf. (A.1-5). Of course, as n increases, the accuracy of our estimates decreases, since (3.7) becomes progressively cruder as n becomes larger.

Another useful procedure, which gives quick estimates once $\langle E^2 \rangle_{xpt}$ and the PD, P_1, have been determined from the data, is to use the empirical observation (from both computation and experiment) that when A_A is 0(0.5 or less), then A_A is approximately

$$A_A \doteq P_1 \text{ at point where sharp rise in } P_{1-A} \text{ vs. } \mathcal{E}_o \text{ occurs.} \qquad \text{(A.1-6a)}$$

cf. Fig. (2.1), (2.2), (3.2)II of Ref. 1. For $0.1 \leq A_A < 0.6$ (approx.) one should pick the ordinate (for P_1) where the departure from the (straight-line) rayleigh portion of the curve begins to occur, cf. Fig. (3.2)II. For $A_A \leq 0.1$, it is the point (i.e. value of P_{1-A}) at which the sharp "rise" in $\mathcal{E}_o$ vs. P_1 takes place. An empirical form of (A.1-6a) is

$$A_A \doteq 10^{-b} = P_1(\text{"jump"}),\ b \leq 0.1. \qquad \text{(A.1-6b)}$$

Similar observation for Γ_A' can be made: at the point where P_{1A} departs (observably) from its straight line, rayleigh behavior, the abscissa ($\mathcal{E}_0$) value yields a good approximation of Γ_A'(db). Coupling this with $\langle E^2 \rangle_{xpt} \doteq 2\Omega_{2A}(1+\Gamma_A')$ then gives us directly

$$(\Gamma_A')_{empir} \doteq \langle E^2 \rangle_{xpt}/2\Omega_{2A} - 1 \quad (>0) \quad , \Gamma_A' \lesssim 0(1) \ . \qquad \text{(A.1-7)}$$

In this way we obtain from $\langle E^2 \rangle_{xpt}$ and P_{1-xpt} what are quite good "start-up" values of $\mathcal{P}_{3A}$, as long as the noise is reasonably nongaussian, i.e. $A_A < 0(0.6)$, $\Gamma_A' < 0(10)$. These results remain to be compared with the parameters obtained from (3.14), according to the procedures outlined above.

A.2 Preliminary Procedures for Parameter Estimation of Class B Noise:

The "start-up" situation for initial parameter estimation in the case of Class B interference is not so direct, however, as in the Class A cases. This stems basically from the fact that two approximating PD's (and c.f.'s and pdf's) are required for an effective (quantitative) description of the first-order statistics (cf. Sec. 4 earlier). A moment approach like that of Sec. 3.2 leading to explicit relations like (3.14), is not analytically direct. In addition, there are now six parameters, $\mathcal{P}_{6B} = [A_B, \Gamma_B', \Omega_{2B}, \alpha, b_{1\alpha}, N_I]$, [cf. Sec. 4, (4.1), rather than the three $\mathcal{P}_{3A}$ $(=A_A, \Gamma_A', \Omega_{2A})$ specifying Class A interference. Nevertheless, with $\langle E^2 \rangle_{xpt}$ and the PD $[=(P_1)_{xpt}]$, and the following empirical observations, we can obtain acceptable "start-up" or initial values, of the defining parameters, before using the mesh or curve-fitting technique, based on an appropriate set of precomputed PD's which employ parameter values in the neighborhood of the initial estimates.

Let us briefly indicate the procedure, when the bend-over point $\mathcal{E}_B$ is empirically known:

(i). $\underline{\Gamma_B', \Omega_{2B}}$: Here we use the observation, similar to the Class A cases (cf. A.1-7), that Γ_B' is quite closely determined when the envelope threshold $\mathcal{E}_0$ is normalized by $\langle E^2 \rangle_{xpt}$), by the abscissa ($\mathcal{E}_0$ in db) corresponding to the point on the PD where the PD begins to depart from rayleigh. For example, we note from Figs. 2.5, Ref. 1; [and from Figs. 2.6, 2.8, Ref. 2, as well as Fig. 2.4, Ref. 1, where now one must use the actual normalization factors therein to correct to a normalization based on $\langle E^2 \rangle_{xpt}$], that $\Gamma_B' = -22$ db ($\approx 8 \cdot 10^{-3}$).

Having estimated Γ_B', we next use

$$\Omega_{2B} \doteq \langle E^2 \rangle_{xpt} / 2(1+\Gamma_B') \qquad \text{(A.1-8)}$$

to get the estimate of Ω_{2B}. Note that when the amount of gaussian component is small, as is usual for moderately to highly nongaussian interference environment, e.g. $\Gamma_B' << 1$), the effect of missestimation of Γ_A' is negligible on Ω_{2B}.

(ii). $\underline{A_B}$: To estimate the structure factor, or impulsive index, A_B, we see here, also, that an empirical relation similar to (A.1-6a,b) holds, provided we choose as the defining ordinate (P_{1B}) that which coincides with the turn-over point ε_B. [In conjunction with our previous observation, (A.1-6), on Class A noise, this is suggested by the fact that a Class A form is used for P_{1-B-II}, cf. Sec. 4 above, and Sec. 3.2.1, Fig. 3.5(II) of Ref. 1.] Thus, we have here

$$A_B \doteq P_1(\mathcal{E}_o \geqq \varepsilon_B) \qquad \text{(A.1-9)}$$

for a reasonable "start-up" value. Again, Figs. 2.4, 2.5 (Ref. 1) and others, bear this out consistently.

(iii). α: the spatial density-propagation parameter α [cf. (5), Sec. 2.3, (2.37), Ref. 1], ($0 < \alpha < 2$), is more elusive. However, we recall that α is one of the "shape" or form parameters: larger α's ($\geq O(1)$) lead to more gradual rises to the bend-over point $\mathcal{E}_B$, [cf. Figs. 2.4, 2.5; also Figs. 3.3, 3.4, Ref. 1]. We may then expect that some measure of the slope of P_{1-B} (vs. ε_o) may yield an estimate of α. This indeed turns out to be the case: we have found that the relation

$$\alpha \doteq \{[P_1(\mathcal{E} \geq \mathcal{E}_o = \mathcal{E}_B), \text{ in db}] - [P_1(\mathcal{E} \geq \mathcal{E}_o = 0\text{db}), \text{ in db}]\}/(-10)(\text{db}), \qquad \text{(A.1-10)}$$

gives a surprisingly accurate estimate, within O(10%). Again, this has been tested on the results of Figs. 2.4, 2.5 2.5 (Ref. 1), and the calculated P_{1-B} of Ref. 2. (Sec. 3).

(iv). G_B: With Γ'_B, A_B, and α in hand [(i)-(iii)], the resulting estimate of the scaling factor G_B is readily made from the relation

$$G_B = \frac{1}{2}\{(\frac{4-\alpha}{2-\alpha} + \Gamma'_B)/(1+\Gamma'_B)\}^{1/2} . \qquad \text{(A.1-11)}$$

Our reason for determining G_B, which is not an independent, or basic, parameter of the defining set $\mathcal{P}_{6B}$, stems from its relation to the normalization $\hat{\mathcal{E}}_o = \mathcal{E}_o(N_I/2G_B)$, used for P_{1-B-I}, for the small and intermediate values of $\mathcal{E}_o$, cf. (4.10), (4.10a). Accordingly, from the <u>experimental PD</u>, P_{1-B-I},

(v). <u>N_I:</u> we seek the $N_I/2G_B$ $(\equiv D_B^{-1})$ factor which aligns the theoretical $P_{1-IB} \doteq 1-\hat{\mathcal{E}}_o^2$ with P_{1-xpt} as $\mathcal{E}_o(\hat{\mathcal{E}}_o) \to 0$. This is done by inspection, and thus we may use

$$N_I = 2G_B D_B^{-1} , \tag{A.1-12}$$

where G_B is estimated from (A.1-11), to obtain the desired estimate of the scaling factor N_I.

(vi). <u>$\hat{A}_\alpha$, or A_α $(\sim b_{1\alpha})$:</u> There remains the basic parameter $b_{1\alpha}$ [or $\langle \hat{B}_{oB}^{\alpha} \rangle$, cf. (4.10a), since $b_{1\alpha} = \Gamma(1-\alpha/2)\langle(\hat{B}_{oB}/\sqrt{2})^{\alpha}\rangle \cdot 2^{\alpha/2-1}\Gamma(1+\alpha/2)$, Eq. (2.38a), Ref. 1.], or $A_\alpha \equiv 2^{\alpha} b_{1\alpha} a_B^{\alpha} A_B$, cf. (3.6a) Ref. 1. For these we use the expansion of (4.10) when $\hat{\varepsilon}_o \doteq 0$, e.g.,

$$\hat{A} \doteq [1- \frac{1-(P_1)_{xpt}}{\hat{\mathcal{E}}_o^2}] \frac{1}{\Gamma(1+\alpha/2)} \ (>0), \tag{A.1-13}$$

where $\hat{\mathcal{E}}_o = \mathcal{E}_o D_B^{-1}$ is given above by the procedure leading to (A.1-12), with $\mathcal{E}_o \equiv E_o/\langle E^2 \rangle_{B-xpt}$, as before. Since

$$\hat{A} \equiv A_\alpha / 2^{\alpha} G_B^{\alpha}, \tag{A.1-14}$$

c.f.(4.10a), and we have found α, G_B above, we now have the desired estimate of A_α, from $\hat{A}$, (A.1-14), which in turn, can give us $b_{1\alpha}$.

(vii). <u>$\mathcal{E}_B$ not known:</u> Finally, when $\mathcal{E}_B$ is unavailable, we can get D_B, $\hat{A}$ from (A.1-12,13). For estimates of the other parameters (α, Γ_B', A_B, Ω_{2B}) the most direct procedure is to postulate a bend-over point $\mathcal{E}_B$, and proceed as in (i)-(v) above. Noticeable error can be involved, however, so that curve-fitting should be invoked, to establish better estimates.

This completes the preliminary estimation procedure for the Class B parameters, giving us reasonably close estimates about which the more refined comparisons with computed PD's enable us to select more precise values, if needed. [It should be noted that the paramters of the theoretical comparisons with experiment, Refs. 1 and 2 were obtained originally by a curve-fitting comparison with calculations using a mesh of parameter values, and some of the "empirical" insights described above here.]

A.3 Remarks on Degree of "Nongaussianness":

In addition to providing us with comparatively quickly determined parameter estimates, these quasi-empirical procedures give a good indication of the extent to which the nongaussian character of the interference is related to the values of the basic parameters $\mathcal{P}_{3A}$, $\mathcal{P}_{6B}$. Of course, given an experimental PD the nongaussian effect is at once generally apparent by the extent to which the "rare events" depart from the (rayleigh, or gauss) line, and where, i.e. at what value of P_1, this occurs. In more detail, however, the parameter estimates help quantify this effect:

For Class A noise, a significant nongauss character is exhibited if $\Gamma'_A < 1$ and the Impulsive Index, A_A is reasonably small, e.g. $A_A \leq 2,3$, say. A highly nongaussian nature is exhibited if A_A is $O(10^{-1})$ or less and $\Gamma'_A < 0.1$: as expected, there is little gauss component and the nongaussian portion is highly structured (small A_A).

Similarly, for Class B noise, its character will be strongly nongaussian if A_B (or A_α, $\hat{A}$) is small, $O(1)$ or less, and Γ'_B is likewise small, $O(\leq 10^{-1})$. Small α $(0 < \alpha < 1)$ leads to a sharper rise (in $\mathcal{E}_0$ vs P_1, cf. Fig. 2.4, Ref. 1, for example) than does larger α $(1 < \alpha < 2)$, cf. Fig. 2.5, Ref. 1, but does not appear to affect strongly the degree of nongaussianness, which is primarily established by A_B (or A_α, $\hat{A}$) and Γ'_B.

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Glossary of Principal Symbols

A.	A_o	= peak amplitude of tppical input signal
	A_A, A_B	= Impulsive Indexes, (Class A,B interference)
	A_α	= effective Impulsive Index (Class B)
	a_A, a_B, a	= normalizing factors
	APD	= a posteriori probability; here 1-Distribution=P_1
	ARI	= combined aperture-RF-IF receiver input stages
	$\mathcal{A}_T, \mathcal{A}_R$	= source, receiver beam patterns
	α	= spatial density-propagation parameter
	α_o	= significance level of test of hypothesis, H_o; (type I error probability)
B.	$B_o, \hat{B}_{oA}, \hat{B}_{oB}$	= generic or typical envelope of waveform from ARI receiver stage
	$b_{1\alpha}, b_{2\alpha}, b_{(2\ell+2)\alpha}$	= weighted moments of the generic envelope $\hat{B}_{oB}$
C.	c.f.	= characteristic function
D.	D_1	= probability distribution
	δ	= delta (singular) function
E.	E, E_o	= instantaneous envelope
	$e_{o\gamma}$	= limiting receiver voltage
	$\mathcal{E}, \mathcal{E}_o, \mathcal{E}_o', \mathcal{E}_o', \mathcal{E}_o, \hat{\mathcal{E}}_{o1,o2}$	= normalized (instantaneous) envelopes; $\mathcal{E}_o$=envelope threshold
	$\mathcal{E}_B$	= "bend-over" point (Class B), empirical pt. of inflexion in P_{1-B}
	e	= $E - \langle E \rangle$
F.	$\hat{F}_1, F_1$	= characteristic functions
	${}_1F_1$	= confluent hypergeometric function
	$\Delta f_N (\Delta f_{ARI} = \Delta f_R)$	= noise, receiver bandwidths
	f	= frequency

G.	G_o	= a basic waveform
	$g(\underset{\sim}{\lambda})$	= geometrical factor of received waveform
	Γ'_A, Γ'_B	= ratio of (intensity of) gaussian component to that of the "impulsive", or nongaussian component
	$\Gamma(x)$	= gamma function
	$\gamma(Y\|\underset{\sim}{X})$	= estimator of Y, given $\underset{\sim}{X}$.
H.		
I.	$\hat{I}_T, \hat{I}$	= exponent of characteristic function
	I_c	= incomplete Γ-function
	$\hat{i}_R$	= unit vector
J.	J_o,	= Bessel functions, 1st-kind, (0 order).
L.	Λ	= domain of integration
	λ	= argument of the c.f.
	$\underset{\sim}{\lambda}$	= (λ,θ,ϕ), coördinates
N.	n.b.	= narrow-band
	N_I	= scaling parameter
O.	Ω_{2A}, Ω_{2B}	= mean intensity of the nongaussian component
	ω, ω_o	= angular frequencies (ω_o=carrier angular fr.)
	$\Omega_{2k,A,B}$	= scale parameters of Class A,B models
P.	$P_1, \hat{P}_1, P_{1-I,II}$	= APD or exceedance probabilities
	pdf	= probability density function
S.	$\sigma, \sigma_G, \hat{\sigma}, \sigma_{mA,B}, \Delta\sigma_G^2, \sigma_\Lambda, \sigma_R^2, \sigma_E^2$	= variances

T. t, t_1, t_2 = times

$\underset{\sim}{\theta}, \underset{\sim}{\theta}'$ = sets of waveform parameters

W. $w_1, W_1, \hat{w}_1$ = probability density functions

X. X = instantaneous amplitude

Z. z_o = a normalized time

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15. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

Methods of determining the first-order parameters of both the approximate and exact Class A and B noise models are derived, for both the ideal case of infinite sample data and the practical cases of finite data samples. It is shown that all (first-order) parameters of these models can, in principle, be obtained, exactly or approximately, from the ideal or practical measurements. [All first-order parameters of Class A but only the (first-order) even moments of the Class B models are exactly obtainable in the ideal case.] Procedures for establishing meaningful measures of the accuracy of the parameter estimates in the practical cases are identified: suitably adjusted, non parametric, small-sample tests of "goodness-of-fit" (such as Kolmogorov-Smirnov tests) provide the principal techniques for establishing accuracy, at an appropriately selected significance level (α_0). The roles of robustness and stability of the parameters and their estimators are also discussed.

16. Key Words *(Alphabetical order, separated by semicolons)*

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