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STATISTICAL-PHYSICAL MODELS OF MAN-MADE AND NATURAL RADIO NOISE

PART II: FIRST ORDER PROBABILITY MODELS OF THE ENVELOPE AND PHASE

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PREFACE

This Report is the second (i.e. Part II) in a series of ongoing studies [Middleton, 1974] of the general electromagnetic (EM) interference environment arising from man-made and natural EM noise sources, and is also part of the continuing analytical and experimental effort whose general aims are [Spaulding and Middleton, 1975]:

(1). to provide quantitative, statistical descriptions of man-made and natural electromagnetic interference (as in this series);

(2). to indicate and to guide experiment, not only to obtain pertinent data for urban and other EM environments, but also to generate standard procedures and techniques for assessing such environments;

(3). to determine and predict system performance in these general electromagnetic milieux, and to obtain and evaluate optimal system structures therein, for

   (a). the general purposes of spectrum management;

   (b). the establishment of appropriate data bases thereto; and

   (c). the analysis and evaluation of large-scale telecommunication systems.

With the aid of (1) and (2) one can predict the interference characteristics of selected regions of the electromagnetic spectrum, and with the results of (3), rational criteria of performance can be developed to predict the successful or unsuccessful operation of telecommunication links and systems in various classes of interference. Thus, the combination of the results of (1)-(3) provide specific, quantitative procedures for spectral management, and a reliable technical base for the choice and implementation of policy decisions thereto.

The man-made EM environment, and most natural EM noise sources as well, are basically "impulsive", in the sense that the emitted waveforms have a highly structured character, with significant probabilities of large interference levels. This is noticeably different from the usual normal (gaussian) noise processes inherent in transmitting and receiving elements. This highly structured character of the interference can drastically degrade performance of conventional systems, which are optimized, i.e. designed to operate most effectively, against the customarily assumed normal background noise processes. The present Report is devoted to the problems of (1), (2) above, namely, to provide adequate statistical physical models, verified by experiment, of these general "impulsive", highly non-gaussian interference processes, which constitute a principal corpus of the interference environment, and which are required in the successful pursuit of (3), as well. The principal new results here are:

(i). Canonical analytical models, experimentally corroberated, for the first-order statistics of the envelope and phase of Class A and Class B noise*;

(ii). Procedures for estimating the (canonical) model parameters, calculation of moments, APD's (= exceedance probabilities, $P_1(\mathcal{E}>\mathcal{E}_0)$, etc.) and probability density functions (pdf's), and a variety of other pertinent statistics; [see the Table of Contents].

Finally, we emphasize, again, that it is the quantitative interplay between the experimentally established, analytical model-building for the electromagnetic environment, and the evaluation of system performance therein, which provides essential tools for prediction and performance, for the development of adequate, appropriate data bases, procedures for effective standardizations, and spectrum assessment, required for the effective management of the spectral-use environment.

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* Class A and Class B noise are distinguished, qualitatively, by having input bandwidths which are respectively narrower and broader than that of the (linear) front-end stages of the typical (narrow-band) receiver in use. More precise definitions are developed in the text following.

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Most man-made and natural electromagnetic interferences are highly non-gaussian random processes, whose degrading effects on system performance can be severe, particularly on most conventional systems, which are designed for optimal or near optimal performance against normal noise. In addition, the nature, origins, measurement and prediction of the general EM interference environment are a major concern of an adequate spectral management program. Accordingly, this second study in a continuing series [cf. Middleton, 1974] is devoted to the development of analytically tractable, experimentally verifiable, statistical-physical models of such electromagnetic interference.

Here, classification into three major types of noise is made: Class A (narrowband vis-à-vis the receiver), Class B (broadband vis-à-vis the receiver), and Class C (=Class A+Class B). First-order statistical models are constructed for the Class A and Class B cases. In particular, the APD (a posteriori probability distribution) or exceedance probability, PD, viz. $P_1(\mathcal{E}>\mathcal{E}_0)_{A,B}$, and the associated probability densities, pdf's, $w_1(\mathcal{E})_{A,B}$, of the envelope are obtained; [the phase is shown to be uniformly distributed in $(0,2\pi)$]. These results are canonical, i.e., their analytic forms are invariant of the particular noise source and its quantifying parameter values, levels, etc. Class A interference is described by a 3-parameter model, Class B noise by a 6-parameter model. All parameters are deducible from measurement, and like the APD's and pdf's, are also canonical in form: their structure is based on the general physics underlying the propagation and reception processes involved, and they, too, are invariant with respect to form and occurrence of particular interference sources.

* The title of this and succeeding Reports in this series, is modified slightly, to emphasize the scope of application, which includes natural as well as man-made interference.

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Excellent agreement between theory and experiment is demonstrated, for many types of EM noise, man-made and natural, as shown by a broad spectrum of examples. Results for the moments of these distributions are included, and more precise analytical conditions for distinguishing between Class A,B, and C interference are also given. Methods for estimating the canonical model parameters from experimental data (essentially embodied in the APD) are outlined in some detail, and a program of possible next steps in developing the theory of these highly nongaussian random processes for application to general problems of spectrum management is presented.

Key Words: Man-made radio noise, Radio noise models, Statistical communication theory.

STATISTICAL-PHYSICAL MODELS OF MAN-MADE AND NATURAL* RADIO NOISE PART II: FIRST-ORDER PROBABILITY MODELS OF THE ENVELOPE AND PHASE

by

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PART I: INTRODUCTION, RESULTS AND CONCLUSIONS

1. INTRODUCTION

As in previous studies (for example, [Middleton, 1972a, 1972b, 1973, 1974]) our central problem is to construct analytically tractable models of man-made and natural radio noise. This is done for three principal technical purposes:

(i). to provide realistic, quantitative descriptions of man-made and natural electromagnetic (EM) interference environments;

(ii). to specify and guide experiments for measuring such interference environments; and,

(iii). to determine the structure of optimal communication systems and to evaluate and compare their performance with that of specified, suboptimum systems, when operating in these general classes of EM interference.

These three tasks, in turn, are critical elements in any adequate program of spectrum management [for example, [Middleton, 1975a].

Our aim here, then, as earlier in this series [cf. Part I, Middleton 1974] is to provide analytical models (1), which combine the appropriate physical and statistical descriptions of general EM interference

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* The title of this, and succeeding Reports in this series, is modified slightly, to emphasize the scope of application, which includes natural as well as man-made interference. Similar remarks apply to the initial study in this series (Part I: OT Report 74-36, April, 1974).

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environments; (2), which are analytically manageable; (3), which possess general, canonical properties - i.e., are not specialized to individual noise mechanisms, source distributions, and emission waveforms, for example; and most important, (4), which are both experimentally verifiable and predictive. In addition, the basic, or "generic" parameters of such statistical-physical models must be measurable quantitites with specified physical structure and interpretation. To achieve this is clearly a nontrivial problem, mainly because of the inherent, highly nongaussian nature of these random processes, a characteristic which at once predicates complex descriptions, and resulting difficulties for the analysis of system performance. That these difficulties can be effectively overcome for model-building (i), and experimental verification (ii) will be evident from the results and analyses in this report (also [Middleton, 1974]). For receiver design and performance (iii), this has already been established by recent work of Spaulding and Middleton [1975].

1.1 Classification of EM Interference:

General EM interference environments can be conveniently classified into three broad categories of interference vis-à-vis any narrow-band* receiver:

Class A Interference: This noise is typically narrower spectrally than the receiver in question, and as such generates ignorable transients in the receiver's front-end (i.e., initial linear stages, viz. aperture-RF-IF) when a source emission terminates;

Class B Interference: Here the bandwidth of the incoming noise is larger than that of the receiver's front-end stages, so that transient effects, both in the build-up and

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* This can be broadened to include receivers of arbitrary bandwidth. However, for almost all EM applications narrow band receivers (i.e., those for which the bandwidth of the initial, linear stages is much less than the RF (and IF) central frequencies) are used exclusively. Henceforth here we shall accordingly consider only narrowband receivers. (The IF-stage is regarded as linear, as far as the narrow-band input is concerned, i.e., heterodyning from RF to IF frequency is linear in the input wave.)

decay occur, with the latter predominating. The receiver is to varying degrees "shock-excited", particularly for inputs of very short duration, so that the receiver is said to "ring".

Class C Interference: This is the sum of Class A and Class B interference, which can occur either because of the presence of sources of mixed types (producing Class A, Class B emissions vis-à-vis the receiver), and/or because any received emission is itself strictly Class C: there is always a build-up interval and a decaying transient period in any receiver front-end reaction to an incoming emission. Effective Class C occurs in this latter instance where the build-up and decay times (at comparable levels) are themselves comparable.

For Class A noise the transient decay period is negligible vis-à-vis the emission's duration, while for Class B interference it is highly dominant. See, for example, Fig. (2.1), Part II, following. [More precise, quantitative conditions specifying Class A, or Class B types, vis-à-vis Class C and each other, are derived in Section 7, Part II.]

The above three categories for interference, as it impacts on a typical (narrow-band) receiver, e.g., as (the linear, front-end of) that receiver responds to the EM environment, provide a useful way of describing the different effects which these different categories have on reception. This categorization is useful because receiver response is statistically different for each Class. As will be seen presently, these differences appear most generally and explicitly (as far as first-order statistics are concerned cerned) in the experimentally derived, and theoretically determined exceedance probabilities (PD's) [also often called APD's (a posteriori probability distributions, cf. Spaulding, [1971])], such as $P_1(X>X_0)$, or $P_1(\mathcal{E}>\mathcal{E}_0)$, which are the respective probabilities that the instantaneous amplitude, or instantaneous envelope observed at the receiver's IF output exceed some threshold X_0, or $\mathcal{E}_0$, as these latter are allowed to assume values in the interval $(-\infty,\infty)$, or (o,∞). Furthermore, this categorization is recommended because the conditions governing the various Classes are simple to distinguish, cf. remarks in Section 7 (II). The conditions "spectrally broader than", and "spectrally narrower than", cf. Fig. (1.1), are to be interpreted as "sufficiently broader or narrower", etc., where in any case, care is taken to refer to the definitions of Class A,B, etc., in terms of the residual transients vs. the "on"-time of the input emission which appears at the output of the IF stage of the receiver in question.

It is instructive to extend our schema of classification further, in order to distinguish between man-made and natural interference, and between "intelligent" and "nonintelligent" emissions. Accordingly, we define:

(i). "Intelligent" noise or interference as man-made and intended to convey a message or information of some sort; whereas,

(ii). "Nonintelligent" noise or interference may be attributable to natural phenomena, e.g., atmospheric noise or receiver noise, for example, or may be man-made, but conveys no intended communication, such as automobile ignition, or radiation from power lines, etc.

[We remark again [cf. Middleton, 1960, Sec. 1.3-5] that by definition, "noise" or "interference" is any undesired "signal" at or in the receiver, regardless of origin.] The importance of distinguishing man-made from natural noise lies in the fact that the former is potentially controllable, sometimes to the point of elimination, whereas the latter cannot be eliminated, at the source, and is usually not subject to control: one can seek only to investigate its effects on the communication process. Moreover, the distinction between "intelligent" and "nonintelligent" is always significant with regard to information transfer: the taxonomy of the former can have greatly different implications and consequences from that of the latter.

We can readily tabulate these different varieties of interference,

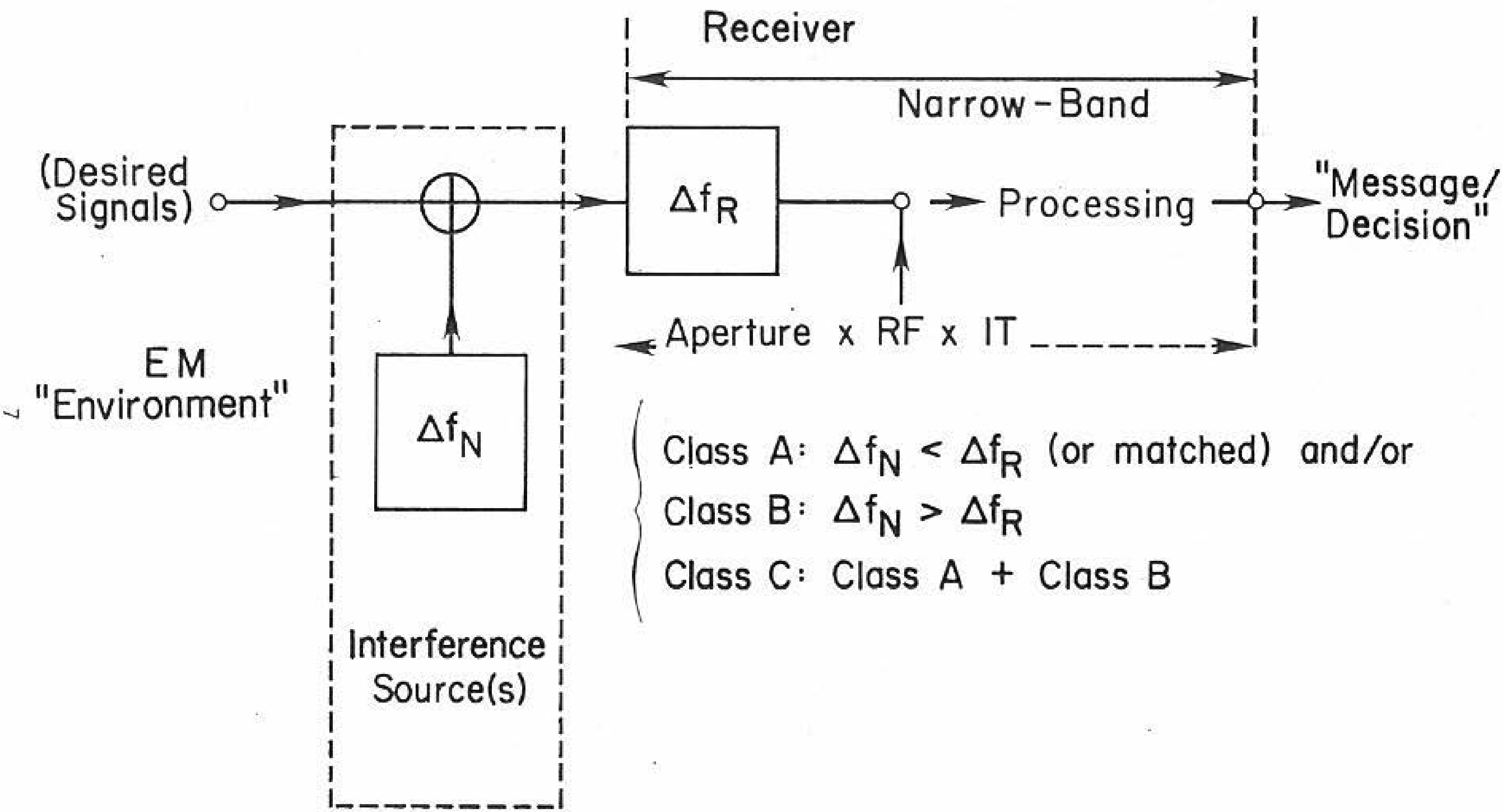

Figure 1.1. Schema of the EM interference and desired signal environment vis-à-vis a typical narrow-band receiver.

in a concise way as suggested in Table (1.1) below:

Table 1.1 Interference Categories* and Classes

Type	"Intelligent"	Class	"Nonintelligent"	Class
Man-Made	1). Compatible 2). Incompatible (Communication) [3). Extra-terrestrial (Communication)	A A,B,C A,B,C]	1). Automobile ignition 2). Other EM emissions: power lines, electric tools, etc.	B A,B,C
Natural			1). Atmospheric 2). Extra-terrestrial solar, galactic, cosmic radiation, etc.	B [A], B,C

* The listing here is not intended to be exhaustive.

We have included a further refinement through the term "compatible". By definition, compatible interference here is one that is appropriately matched spectrally to the receiver band Δf_{ARI}, in the sense of being equivalent to Class A interference vis-à-vis the receiver and occupying a spectral region in Δf_{ARI}, and such as to produce ignorable transients in the ARI-stages. "Incompatible" may mean that $\Delta f_N > \Delta f_{ARI}$ (Class B), or that only a portion of the incident emission is spectrally available to the receiver: Class A again, e.g. $\Delta f_{N\text{-effective}} < \Delta f_{ARI}$, but now the interference is not wholly in the receiver band Δf_{ARI}. Class C above in the Table reminds us that combinations of Class A and B noise can occur, as well.

1.2 Earlier Work:

For the most part, earlier efforts at modelling man-made and natural noise (principally atmospheric noise) have produced a wide variety of analytical results, often with the virtue of mathematical simplicity, but severely limited in usefulness by lack of generality and physical insight, and a concommitant dependence on local, empirical data and circumstances. Two important exceptions to the above are the work of Furutsu and Ishida [1960] on obtaining the APD's (and associated probability densities [pdf's]) of atmospheric noise under rather broad conditions, and the more recent studies of Giordano [1970], and Giordano and Haber [1972], similarly directed to atmospheric noise. Both sets of intestigations, however, are (necessarily) constrained to Class B types of interference (cf. Sec. (1.1) above), and neither attempts the canonical formulation, which is a key feature of our current efforts [Middleton, 1972, 1973, 1974, and this Report; see also comments, Sec. 5.3 (II)] following. This canonical formulation allows us to apply the new models formally by Class(A,B, etc.) to all types of (EM) interference, unrestricted in general structure by the particular physical mechanism involved. [These latter, of course, determine the generic properties of the model parameters, and must be specifically introduced into model building if the ad hoc and arbitrary empiricism of much of the earlier work is to be avoided.] For a more detailed review of earlier work vis-à-vis this newer approach, see Chapter 2 of Spaulding and Middleton [1975], and references therein.

1.3 New Results:

The principal new results of this study may be briefly introduced here, in contrast to our remarks above on previous work. Here we obtain canonical, analytical, first-order statistical models of both Class A and Class B interference, specifically for the envelope (E) and phase (ψ) of the narrow-band output of the composite aperture-RF-IF stages of a typical receiver. As noted above, these models are based on a general physical mechanism (cf. Section 2, Middleton [1974], for example), providing, among other things, insight into the parameter structure, as well as contributing, in a broad way to the analytical form of the probability distribution (PD's) and probability densities (pdf's) themselves, which are the principal results here. In addition, the general method of approximating the governing (1st-order) characteristic functions (c.f.'s) is described, which enables us to obtain the required canonical structures in tractable analytic forms. These, in turn, give the resulting analytical models their broad applicability, unrestricted by particular physical mechanisms, and, in fact, controlled only by the underlying poissonian postulate of independent source emissions in space and time [cf. Section (2.1) Part II].

Included, also, are specific procedures for determining the model parameters from experimental data, analytical results for the first-order, Class A and B moments of the envelope, and detailed, quantitative conditions for specifying Class A or Class B interference. Excellent agreement with experiment is found, and a variety of comparisons of theory with experiment is included, involving many different physical types of radio interference, not only to illustrate this agreement, but to demonstrate the canonical character of the approach as well, cf. Section (2.4). Finally, the definition of Class A models, and their quantitative identification with observed noise processes are new, although, of course, such interference has been physically present for many years. Class B models are "classical", although not so designated until now, but here, again, our present approach is to a large extent original, particularly with regard to canonical results.

1.4 Organization of the Report:

As one can see from the Table of Contents, this Report is divided into two principal units: Parts I and II. Part I contains introductory, background material (Section 1), and in Section 2 following an extensive summary and discussion of the main results, as well as related matters and next steps in our program of interference modelling. Part II, on the other hand, is devoted to the detailed analytical development of the theory: Section 2(II) describes the canonical approximations of the characteristic function (c.f.) required for the Class A and B envelope distributions. Sections 3(II) and 4(II) are devoted to these distributions and distribution densitees, while Section 5(II) contains results for the (first-order) moments. In Section 6(II) the problem of determining model parameters from experimental data is addressed, again for Class A and B interference. Section

7(II) completes this study with the derivation of analytical conditions which quantitatively determine when a Class A, or Class B model is appropriate.

2. RESULTS AND CONCLUSIONS

Let us now summarize in some detail the principal results of the analysis in Part II of this Report. Accordingly, Section 2 here is organized as follows: first, we briefly consider the first-order results for the phase, ψ, of the narrow-band output of ARI-stages of a typical narrow-band receiver immersed in the general EM interference environment under analysis here [cf. Section (1.1) above]. Next, we present in Section (2.2) various results for the envelope statistics of Class A interference, with a similar presentation in Section (2.3) for the Class B cases. A number of comparisons with experiment are given in Section (2.4), both to demonstrate the canonical character of our models and to exhibit the excellent agreement between theory and experiment which is obtainable by our present analytical approach. Section (2.5) treats the estimation of the physically-based model parameters; Section (2.6) reviews such other results as moments, limiting forms, the existence of Hall models, conditions for the existence of Class A, B, and C noise types, etc. We conclude with remarks in Section (2.7) on uses, advantages, and limitations of these models and outline a number of next steps for their continuing analytical and experimental development.

2.1 Phase Statistics:

In the general case we may use (2.14), Part II, and the relation $w_1(\psi) = \int_0^\infty w_1(E,\psi)dE$, to obtain the pdf, and the APD ($= \int_\psi^\infty w_1(\psi)d\psi$, $(0 \leq \psi < 2\pi)$), of the instantaneous phase ψ, which will not generally be uniform [on $(0,2\pi)$]. However, in the truly narrow-band situation of Section (2.2)II, we obtain the well-known uniform pdf [(2.21), Part II), e.g. $w_1(\psi)=1/2\pi$, $(0,2\pi)$] as in the simpler, gaussian examples. [Higher-order statistics of ψ, on the other hand, are nonuniform and analytically much more complex: vide Section 9.1.2 of Middleton [1960] in the gaussian cases.] Because of this first-order simplicity for the statistics of the phase we accordingly concentrate our attention on the (first-order) statistics of the associated envelope, which, as expected, departs radically from a gaussian (i.e. rayleigh) behaviour, as our results following, and experiment as well, amply demonstrate.

2.2 Envelope Statistics: The APD and pdf for Class A Interference:

Our principal analytical results here are: (i), the c.f.; (ii), the PD, or exceedance probability $P_1(\mathcal{E}>\mathcal{E}_0)$; and (iii), the associated pdf, $w_1(\mathcal{E})$. These are respectively*

$$\underline{\text{(c.f.):}} \quad \hat{F}_1(ia\lambda)_A \doteq e^{-A_A}\sum_{m=0}^{\infty}\frac{A_A^m}{m!}\,e^{-\hat{\sigma}_{mA}^2 a^2\lambda^2/2} \quad , \; a^2 = [2\Omega_{2A}(1+\Gamma_A')]^{-1} \;\text{cf. Eq. (3.3), Part II} \Big\}, \tag{2.1}$$

with $2\hat{\sigma}_{mA}^2 = (m/A_A+\Gamma_A')/(1+\Gamma_A')$, cf. (Eq. (3.5), Part II; and

$$\underline{\text{(PD):}} \quad P_1(\mathcal{E}>\mathcal{E}_0) \simeq e^{-A_A}\sum_{m=0}^{\infty}\frac{A_A^m}{m!}\,e^{-\mathcal{E}_0^2/2\hat{\sigma}_{mA}^2} \quad , \quad (0 \leq \mathcal{E}_0 < \infty), \tag{2.2}$$

cf. Eqs. (3.7a,b), Part II; and

$$\underline{\text{(pdf):}} \quad w_1(\mathcal{E})_A \simeq e^{-A_A}\sum_{m=0}^{\infty}\frac{A_A^m\,\mathcal{E}e^{-\mathcal{E}^2/2\hat{\sigma}_{mA}^2}}{m!\;\hat{\sigma}_{mA}^2} \quad , \quad (0 \leq \mathcal{E} < \infty) \;, \tag{2.3}$$

cf. Eq. (4.2), Part II. Various curves of P_1, w_1 are given in Figs. (3.1), (3.2), (4.1), (4.2), Part II, showing typical behaviour for selected values of the (global) parameters (A_A, Γ_A'). Here $\mathcal{E}, \mathcal{E}_0$ are normalized envelopes

$$\mathcal{E} \equiv E\Big/\sqrt{2\Omega_{2A}(1+\Gamma_A')} \quad ; \quad \mathcal{E}_0 \equiv E_0\Big/\sqrt{2\Omega_{2A}(1+\Gamma_A')}, \tag{2.4}$$

cf. Eq. (3.1), Part II, where E_0 is some preselected threshold value of the envelope E. (Note that our normalization introduces a third parameter Ω_{2A}.)

The parameters $(A_A, \Gamma_A', \Omega_{2A})$ which appear directly in our statistical results for P_1, w_1 we call <u>"global" parameters.</u> The physical significance of these global parameters $(A_A, \Gamma_A, \Omega_{2A})$ is briefly stated:

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* See the glossary of principal symbols, at the end of the Report.

1). A_A = the Impulsive Index (for Class A interference): this is defined as the average number of emission "events" impinging on the receiver in question times the mean duration of a typical interfering source emission [cf. Eqs. (2.38), (2.39), Part II and associated discussion]. The smaller A_A, the fewer such events and/or their duration, so that the noise properties are then dominated by the waveform characteristics of a typical event. Loosely speaking, we say that such noise is "impulsive", although here the mean duration of events is sufficiently long to avoid generating noticeable transients in the receiver, i.e. we have Class A noise, as defined above, Section (1.1). As A_A is made large, one approaches gaussian (or in the case of the envelope here), rayleigh statistics (cf. Sec. (2.4), Part II).

2). $\Gamma'_A \equiv \sigma_G^2/\Omega_{2A}$ = the ratio of the intensity of the independent gaussian component σ_G^2 of the input interference including received "front-end" noise, to the intensity Ω_{2A} of the "impulsive", non-gaussian (or rayleigh) component, cf. (3.1a), Part II. A portion, σ_E^2, of this normal component [cf. Sec. (2.3.1), Part II] arises from the cumulative effect of a large number of external sources, none of which is so strong as to be considered part of the "impulsive" interference, which is statistically the dominating effect (for small and moderate Indexes, A_A).

3). Ω_{2A} = the intensity of the above-mentioned "impulsive" component, cf. Eq. (3.1a), Part II.

[The rayleigh nature of P_1, w_1 for large Indexes, i.e., when $A_A \to \infty$, is seen at once from (2.2), (2.3), as then $\Gamma'_A \to \infty$ also, so that $2\hat{\sigma}^2_{mA} \to 1$, and

$$\therefore \quad P_1(\mathcal{E}>\mathcal{E}_0)_A \to e^{-\mathcal{E}_0^2} \;;\; w_1(\mathcal{E})_A \to 2\mathcal{E}e^{-\mathcal{E}^2} \quad ,(0\le\mathcal{E}_0,\mathcal{E}\le\infty), \tag{2.5}$$

in normalized form; see Section (2.4), Part II, for the general case.]

Characteristic behaviour of the APD P_{1-A} vs. $\mathcal{E}_0$, cf. Figs. (3.1), (3.2) [and w_{1-A} vs. $\mathcal{E}$], is exhibited by the "rayleigh" form [constant slope $\eta = -2$, on the linear by $-(1/2)\log_{10}(-\log_e[\ ])$ plots of P_{1-A}] for the comparatively small values of threshold $\mathcal{E}_0$, i.e., large values of $P_1(\mathcal{E} > \mathcal{E}_0)_A$, followed by a very steep rise, after which P_{1-A} bends over and approaches some asymptote with fixed slope η, $0 < \eta < 2$, at large $\mathcal{E}_0$ (small P_{1-A}) less than that of the rayleigh behaviour for P_{1-A} in the 0.1-1.0 region. Thus, we have $P_{1-A} \to e^{-a\mathcal{E}_0^{\eta}}$, $\mathcal{E}_0 \to \infty$, $(0<\eta<2)$.

This limiting, finite, and bounded slope as $\mathcal{E}_0$ becomes very large, after the characteristic bend-over, reflects the physical condition that the interference process has finite total <u>average</u> energy; accordingly, no individual source, or finite collection of sources, can emit unbounded energy over any finite period. Furthermore, if the number of sources is <u>finite</u>, with finite <u>power</u>, e.g. no infinite instantaneous amplitudes (as is the case ultimately in practice), then the limiting slope η <u>becomes infinite</u> at some extremely large value of $\mathcal{E}_0$. This effect can show up at comparatively small values of threshold $\mathcal{E}_0$, for example, with a single, finite source, of limited peak emissions, whereas with multiple sources the phenomenon will occur at larger $\mathcal{E}_0$. In any case, these (below-)bounded slopes (>0) insure, also, that all (finite) <u>moments</u> of the envelope exist, as physically required by the condition of finite average emission energy.

In our models, however, we assume that the number of <u>potential</u> emitting noise sources is infinite, although the probability of even a large number radiating at any given instant is very small, according to the fundamental assumption of poissonian "events", e.g., emissions, postulated here. In addition, we permit a distribution of emission levels ($\sim$ amplitude) per source, where infinite magnitudes are possible, similarly with vanishing probabilities of occurrence. Thus, we may expect a nonzero limiting slope for P_{1-A} as $\mathcal{E}_0 \to \infty$: infinite amplitudes can occur, but with vanishing probability*. In practice, however, although the Impulsive Index

*The poissonian "rare-event" dominates any "rare-event" from the gaussian component.

(A_A) may be small, there is always the possibility of the very large amplitude "rare event", i.e. $\mathcal{E} > \mathcal{E}_0$(>>0db). But the practical upper limit on $\mathcal{E}_0$ for such an occurrence is so high, i.e., $P_{1-A} < 0(10^{-8})$ or less, usually, that deviation of the experimental results from those predicted by the theoretical models at these levels has not been practically observed. * For example, see Figs. (2.3)-(2.5) following.

2.3 Envelope Statistics: The APD and pdf for Class B Interference:

The Class B interference requires a more extensive analytical model. This arises because two canonical characteristic functions (c.f.'s) are needed to approximate the exact c.f. [vide Section 2.7, Part II], one for small and intermediate values of the envelope ($0 \leq \mathcal{E} \leq \mathcal{E}_B$), the other for the larger values ($\mathcal{E}_B \leq \mathcal{E}$). The principal analytic results here,are, accordingly:

$$\underline{\text{(c.f.'s):}} \quad \hat{F}_1(ia\lambda)_{B-I} \doteq e^{-b_{1\alpha}A_B a^\alpha \lambda^\alpha - \Delta\sigma_G^2 a^2 \lambda^2/2}, \quad (0 \leq \mathcal{E} \leq \mathcal{E}_B) , \tag{2.6a}$$

$$\hat{F}_1(ia\lambda)_{B-II} \doteq e^{-A_B}\exp[A_B e^{-b_{2\alpha}a^2\lambda^2/2} - \sigma_G^2 a^2 \lambda^2/2], \; (\mathcal{E}_B < \mathcal{E} < \infty), \tag{2.6b}$$

from Eqs. (3.10a,b), Part II, with $a^2 = [2\Omega_{2B}(1+\Gamma_B')]^{-1}$ now, cf. Eq. (3.3), Part II, and

$$\underline{\text{(PD):}} \quad \begin{aligned} P_1(\mathcal{E}>\mathcal{E}_0)_B &\cong P_1(\mathcal{E}>\mathcal{E}_0)_{B-I} , \quad (0 \leq \mathcal{E}_0 \leq \mathcal{E}_B) \\ &\simeq P_1(\mathcal{E}>\mathcal{E}_0)_{B-II} , \quad (\mathcal{E}_B \leq \mathcal{E}_0 < \infty) \end{aligned} \quad . \tag{2.7}$$

Here explicitly we have

---

* We remark, moreover, that our models can be analytically modified to account for a limited, maximum number of emissions at any given instant by truncating the basic summation leading to the usual form [(2.1), Part II] of the characteristic function, cf. Section 2 of Middleton [1974] for details. The functional complexity of the result is, as expected, greatly increased, unless this maximum number is very small.

$$\left\{\begin{array}{l} \hat{P}_1(\hat{\mathcal{E}}>\hat{\mathcal{E}}_0)_{B-I} \equiv \\ P_1(\mathcal{E}>\mathcal{E}_0)_{B-I} \simeq 1-\hat{\mathcal{E}}_0^2 \sum_{n=0}^{\infty} \frac{(-1)^n \hat{A}_\alpha^n}{n!} \Gamma(1+\frac{\alpha n}{2})\,{}_1F_1(1+\frac{\alpha n}{2};2;-\hat{\mathcal{E}}_0^2) \;, \qquad (2.7a) \\ \\ P_1(\mathcal{E}>\mathcal{E}_0)_{B-II} \simeq \frac{e^{-A_B}}{4G_B^2} \sum_{m=0}^{\infty} \frac{A_B^m}{m!} e^{-\mathcal{E}_0^2/2\hat{\sigma}_{mB}^2} \;, \qquad (2.7b) \end{array}\right.$$

with $\hat{A} = A_\alpha/2^\alpha G_B^\alpha$, $\hat{\mathcal{E}}_0 \equiv (\mathcal{E}_0 N_I)/2G_B$, where

$$\left\{\begin{array}{ll} 0 < \alpha < 2; \text{ cf. Eq. (2.82), Part II, et seq.}; & (2.7c) \\ 2\hat{\sigma}_{mB}^2 \equiv (m/\hat{A}_B+\Gamma'_B)/(1+\Gamma'_B),\ \hat{A}_B=(\frac{2-\alpha}{4-\alpha})A_B, \text{ cf. Eq. (3.16a), Part II}; & (2.7d) \\ G_B^2 \equiv 2^{-2}(1+\Gamma'_B)^{-1}(\frac{4-\alpha}{2-\alpha}+\Gamma'_B), \text{ cf. Eq. (3.12b), Part II}. & (2.7e) \end{array}\right.$$

The associated pdf's are (from Eqs. (4.3), (4.4), Part II):

$$\left\{\begin{array}{l} \left.\begin{array}{l} w_1(\mathcal{E})_{B-I} \\ \equiv \hat{w}_1(\hat{\mathcal{E}})_{B-I} \end{array}\right\} \simeq 2\hat{\mathcal{E}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{A}_\alpha^n\, \Gamma(1+\frac{n\alpha}{2})\,{}_1F_1(1+n\alpha/2;1;-\hat{\mathcal{E}}^2),\ 0 \le \mathcal{E} \le \mathcal{E}_B, \qquad (2.8a) \\ \\ w_1(\mathcal{E})_{B-II} \simeq \frac{e^{-A_B}}{4G_B^2} \sum_{m=0}^{\infty} \frac{A_B^m}{m!} \frac{\mathcal{E} e^{-\mathcal{E}^2/2\hat{\sigma}_{mB}^2}}{\hat{\sigma}_{mB}^2} \;, \ (\mathcal{E}_B \le \mathcal{E} < \infty), \qquad (2.8b) \end{array}\right.$$

with ${}_1F_1$, as usual, a confluent hypergeometric function [Middleton, 1960, Appendix A.1.2], so that the $w_1(\mathcal{E})_B = w_1(\mathcal{E})_{B-I}$ for $0 \le \mathcal{E} \le \mathcal{E}_B$, while $w_1(\mathcal{E}_B) = w_1(\mathcal{E})_{B-II}$ when $\mathcal{E} \ge \mathcal{E}_B$. In Part II, Figures (3.6), (3.7) show typical curves of the PD, (2.7), and Figures (4.3), (4.4) for the pdf [(2.8a,b)], for selected parameter values. Again, $\mathcal{E}$, $\mathcal{E}_0$ are normalized according to (3.2), Part II, e.g. like (2.4) above, with Ω_{2A} replaced by Ω_{2B}, etc.

There are now six global parameters for our model: $(A_\alpha, \alpha, A_B, \Gamma'_B, \Omega_{2B}; N_I)$, cf. Section 6B (Part II). The subset $(A_B, \Gamma'_B, \Omega_{2B})$ are, just for Class A interference above, respectively, 1), the Impulsive Index; 2), the ratio of the intensity of the independent gaussian component (σ_G^2) to the intensity of the impulsive component; and 3), the intensity of the impulsive component (Ω_{2B}), itself. These have the physical significance described above in Section 2.2. The additional parameters required here are:

4). $A_\alpha = \frac{2\Gamma(1-\alpha/2)}{\Gamma(1+\alpha/2)} \frac{\langle \hat{B}^\alpha_{oB} \rangle}{\{2\Omega_{2B}(1+\Gamma'_B)\}^{\alpha/2}} A_B$ = an "effective" Impulsive Index proportional to the Impulsive Index A_B, cf. (2.38), (2.39), Part II, which depends on the generic parameter α. Here $\langle \hat{B}^\alpha_{oB} \rangle$ is the α-moment of the basic envelope of the output of the composite ARI stages, cf. Fig. (1.1) above, and Eq. (2.87d) Part II.

5). $\alpha \equiv \left.\frac{2-\mu}{\gamma}\right|_{surface} ; \left.\frac{3-\mu}{\gamma}\right|_{vol}$ = spatial density-propagation parameter, cf. (2.82), Part II et seq.. Here μ, γ are respectively the power law exponents associated with the range dependence of the density distribution of the possibly emitting sources, and their propagation. (See Eq. (2.61) et seq; Section 2.5.2, Eq. (2.63, Part II). The parameter α provides an "effective" measure of the average source density with range. Thus, if we standardize, for example, the propagation law as $\gamma = 1$ (the usual spherical spreading), we have $\mu = 2-\alpha$, $(0<\alpha<2)$, for the source density distribution $\sigma_S \sim \lambda^{-\mu} = \lambda^{\alpha-2}$, ($c\lambda = R$ = distance from a typical source to the receiver). Knowledge of α accordingly gives us a direct measure of effective source density, and if γ is known or measured, separately, then $\mu = 2-\alpha\gamma$ gives us the actual power law for σ_S, cf. (2.63), Part II. [We shall exploit this relationship in detail in a later study in this series.]

6). N_I = The scaling factor which insure that P_{1-I}, w_{1-I} yields the correct mean square envelope $2\Omega_{2B}(1+\Gamma_B')$ (See Sec. (3.2A).

7). $\mathcal{E}_B = \dfrac{E_B}{\sqrt{2\Omega_{2B}(1+\Gamma_B')}}$ = the (normalized) "bend-over" point, at which the two (approximate) forms of PD (and pdf) are joined, according to the procedures discussed in Section 3.2, and Eqs. (3.18)-(3.20), cf. Fig. (3.5), Part II. This is an empirically determined point, representing the point of inflexion (for small P_{1-B}) at which the experimentally determined PD, or exceedance probability $P_1(\mathcal{E}>\mathcal{E}_0)_{B\text{-expt.}}$, bends, e.g. at which $d^2P_{1\text{-expt.}}/d\mathcal{E}_B^2 = 0$. Examples of this are indicated in the next Section, (2.4).

We note that without an (experimental) $\mathcal{E}_B$ we cannot predict the limiting form of the PD as $\mathcal{E}_0 \to \infty$; we can then only obtain the subset of global parameters $(A_\alpha, \alpha ; \Gamma_B', \Omega_{2B}, N_I))$, cf. Section 6C (Part II). Examples of this are the particular cases of atmospheric and automotive ignition noise shown in Figs. (2.3), (2.4), and interference from a fluourescent light, Fig. (2.5).

The six parameters $(A_\alpha, \alpha, A_B, \Gamma_B, \Omega_{2B}', N_I)$ are all physically specified and measurable parameters in the analytical model (provided $\mathcal{E}_B$ is determined). Only $\mathcal{E}_B$ itself is an empirical parameter, without explicit quantitative relationship to the underlying physical mechanisms involved. This is because the simplest canonical approximation to the exact c.f. [Eq. (2.87), Part II] requires a two-part c.f., approximate for one to the small and intermediate values of the envelope, and for the other, to the large values of the envelope. This second c.f., [and PD = $P_1(\mathcal{E} > \mathcal{E}_0)_{B-II}$], provides the needed "bending" of the APD curves for the rare events, as sketched in Fig. (3.5), Part II, for instance, and shown in some of the experimental examples of Section (2.4) following.

Generally, unlike the Class A cases, Class B interference exhibits a much more gradual rise (as $\mathcal{E}_0$ becomes larger), also with increasing α. Similar upward displacement of the rayleigh sections (small $\mathcal{E}_0$) of these APD curves occurs for an increasing gaussian component (Γ_B'), while increasing the Impulsive Index $A_B(\sim A_\alpha)$ also acts to diminish the steepness of these curves as $\mathcal{E}_0$ is increased. The physical necessity for a suitable "bend-over" at the larger values of $\mathcal{E}_0$ has already been discussed above in Section (2.2) for the Class A noise: a fixed, asymptotic slope (η>0) is required, to insure the existence of all moments, which in turn is demanded by the condition of finite total average energy. Again, increasing the Impulsive Index and/or increasing the independent gaussian component (σ_G^2) eventually yields a wholly gaussian process (rayleigh, of course, in the envelope), as expected.

2.4 Comparisons with Experiment:

In this subsection we include a variety of comparisons of our theoretical models with experiment, for both Class A and Class B interference, cf. Figs. (2.1)-(2.8) following. Four significant features are at once evident:

(1). The agreement between theory and experiment is excellent, i.e., the approximating forms are effective, analytical relations for predicting the desired first-order statistics;

(2). The canonical nature of our models is demonstrated: the <u>form</u> of the results [here APD's: $P_1(\mathcal{E}>\mathcal{E}_0)$,], is invariant of the specific source mechanism, whether ignition noise, atmospherics, flourescent light, etc., man-made or natural, within the distinct Class A, or B;

(3). Class A and Class B interference are observeably and quantitatively different noise types (vis-à-vis the narrow-band receiver used).

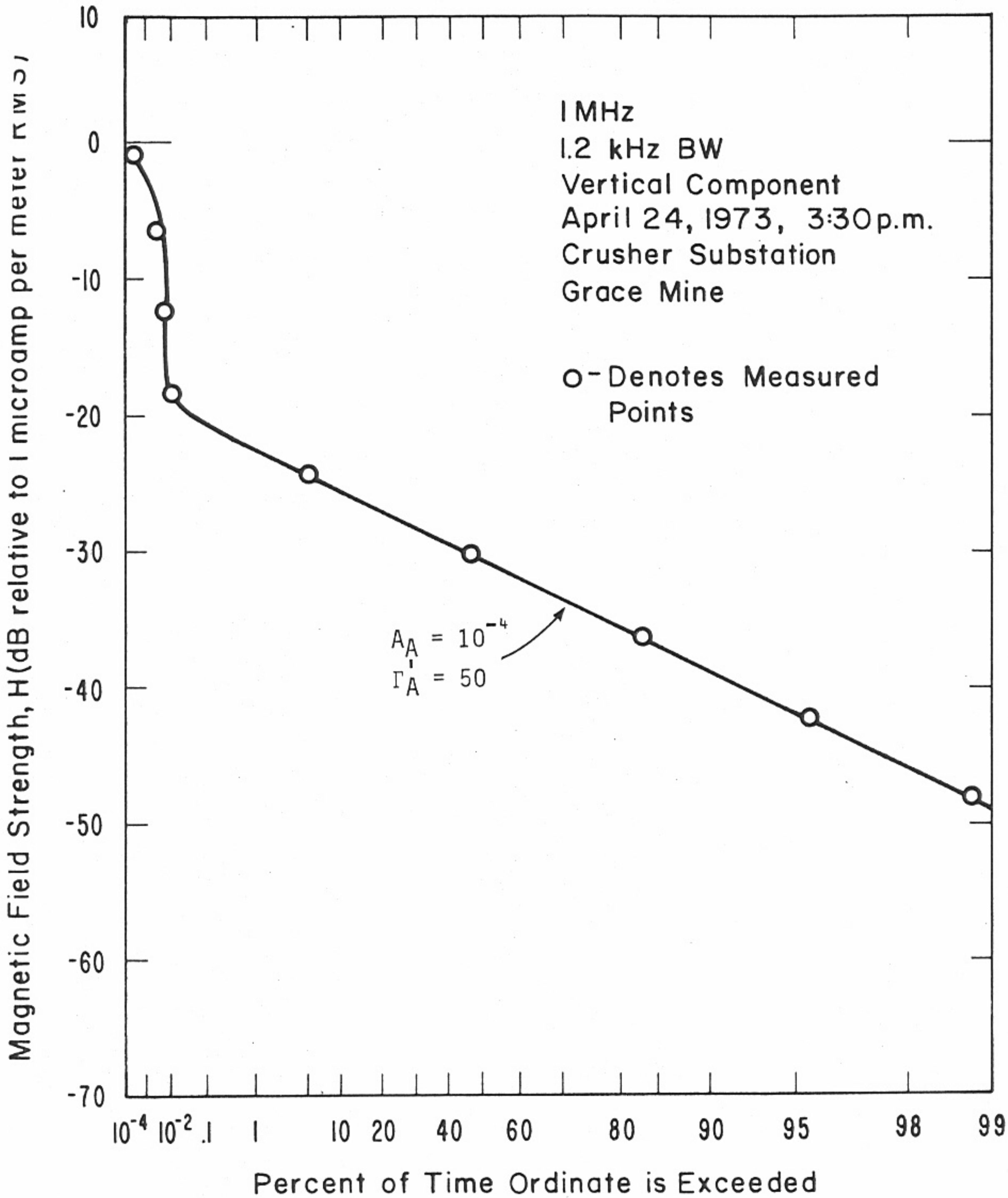

Figure 2.1. Comparison of measured envelope distribution, $P_1(\mathcal{E} > \mathcal{E}_0)_A$, with Class A model, cf. (2.2). Interference from ore-crushing machinery [data from Adams et al (1974)].

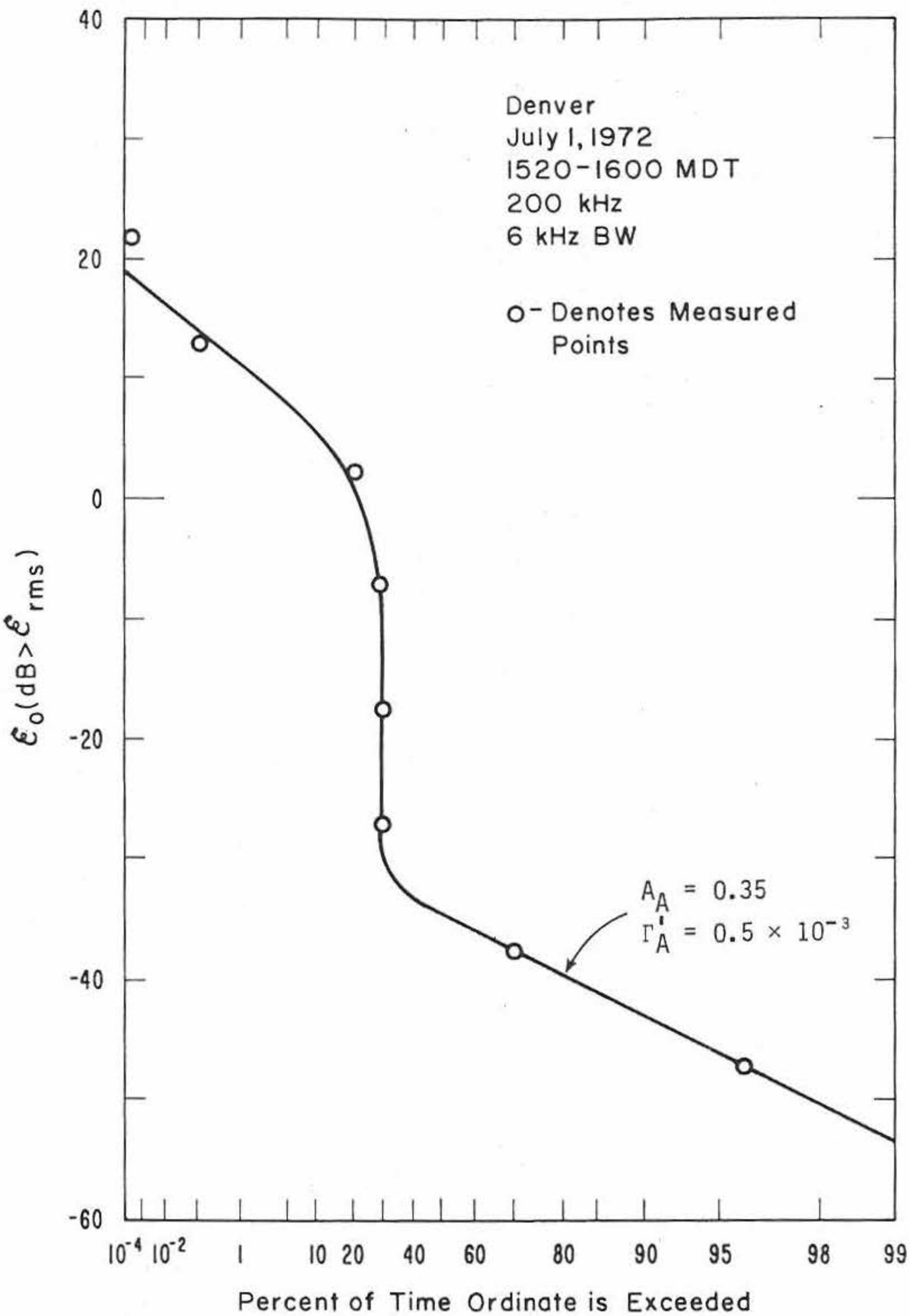

Figure 2.2. Comparison of measured envelope distribution, $P_1(\mathcal{E} > \mathcal{E}_0)_A$, with Class A model, cf. (2.2). Interference (probably) from nearby powerline, produced by some kind of equipment fed by the line [data from Bolton (1972)].

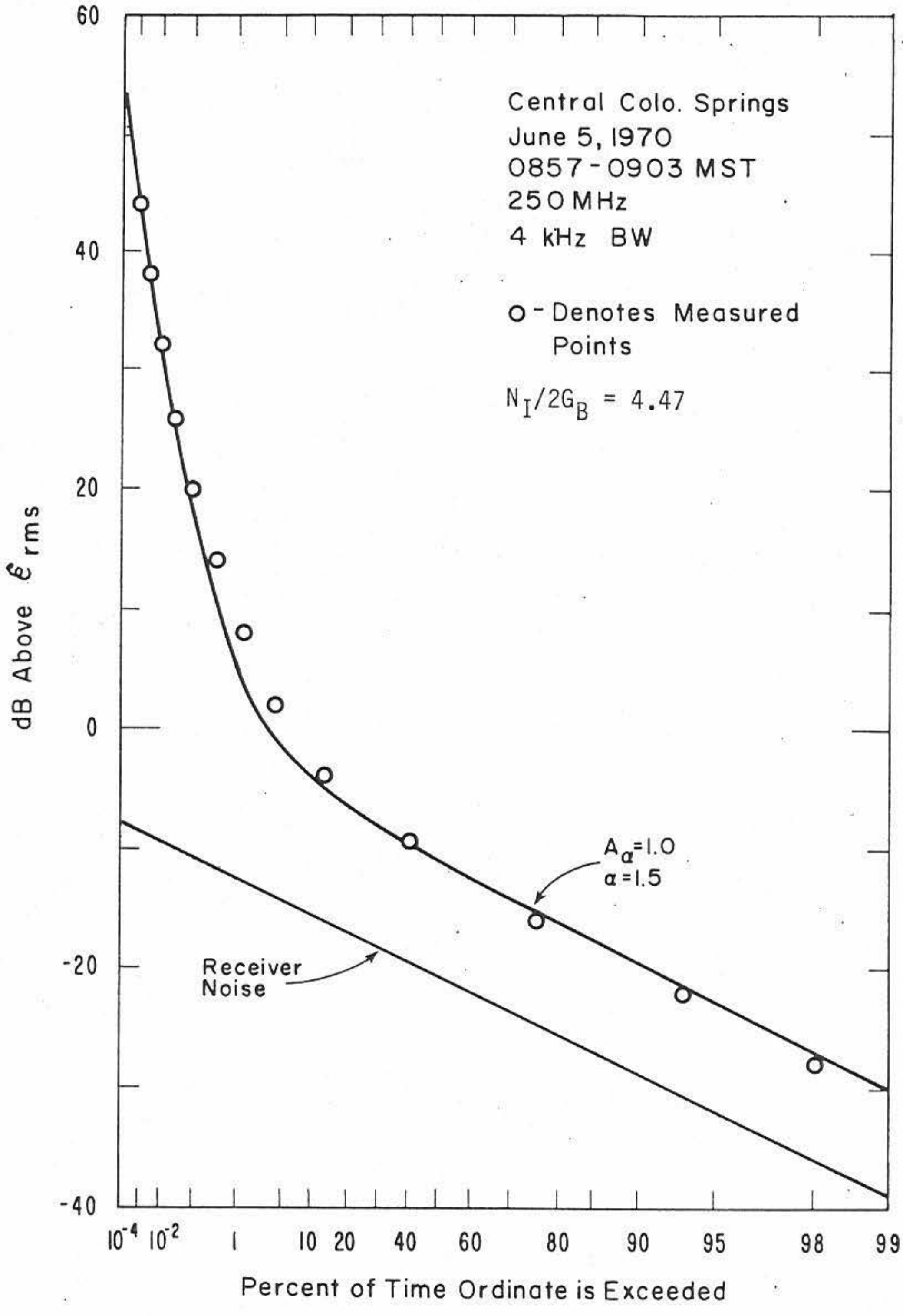

Figure 2.3. Comparison of measured envelope distribution, $P_1(\mathcal{E} > \mathcal{E}_0)_B$, of man-made interference (primarily automotive ignition noise) with Class B model, cf. (2.7a). [Data from Spaulding and Espeland (1971).]

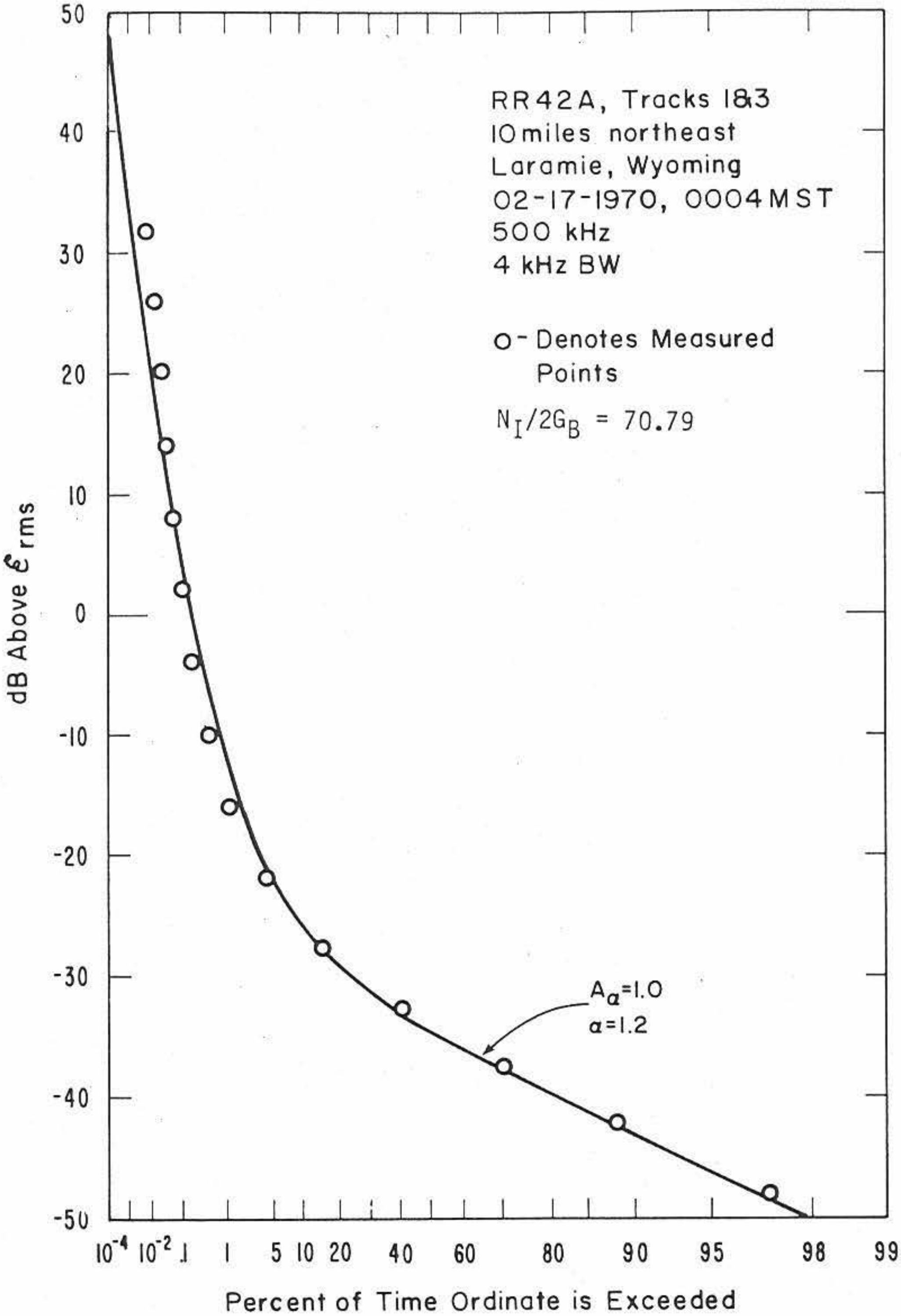

Figure 2.4. Comparison of measured envelope distribution, $P_1(\mathcal{E} > \mathcal{E}_0)_B$, of atmospheric noise with Class B model, cf. (2.7a). [Data from Espeland and Spaulding (1970).]

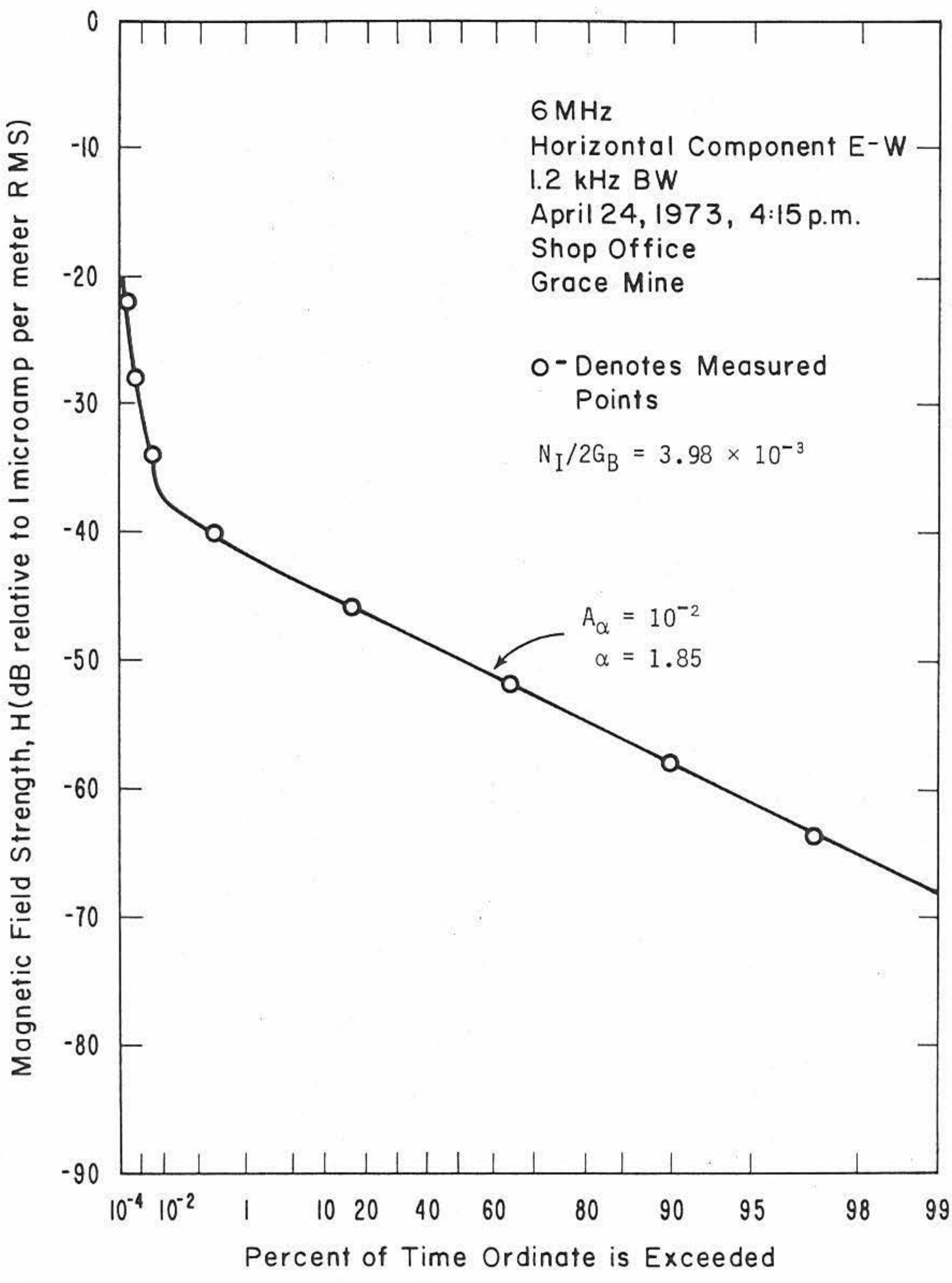

Figure 2.5. Comparison of measured envelope distribution, $P_1(\mathcal{E} > \mathcal{E}_0)_B$ of man-made interference (fluorescent lights in mine shop office) with Class B model, cf. (2.7a). [Data from Adams et al. (1974).]

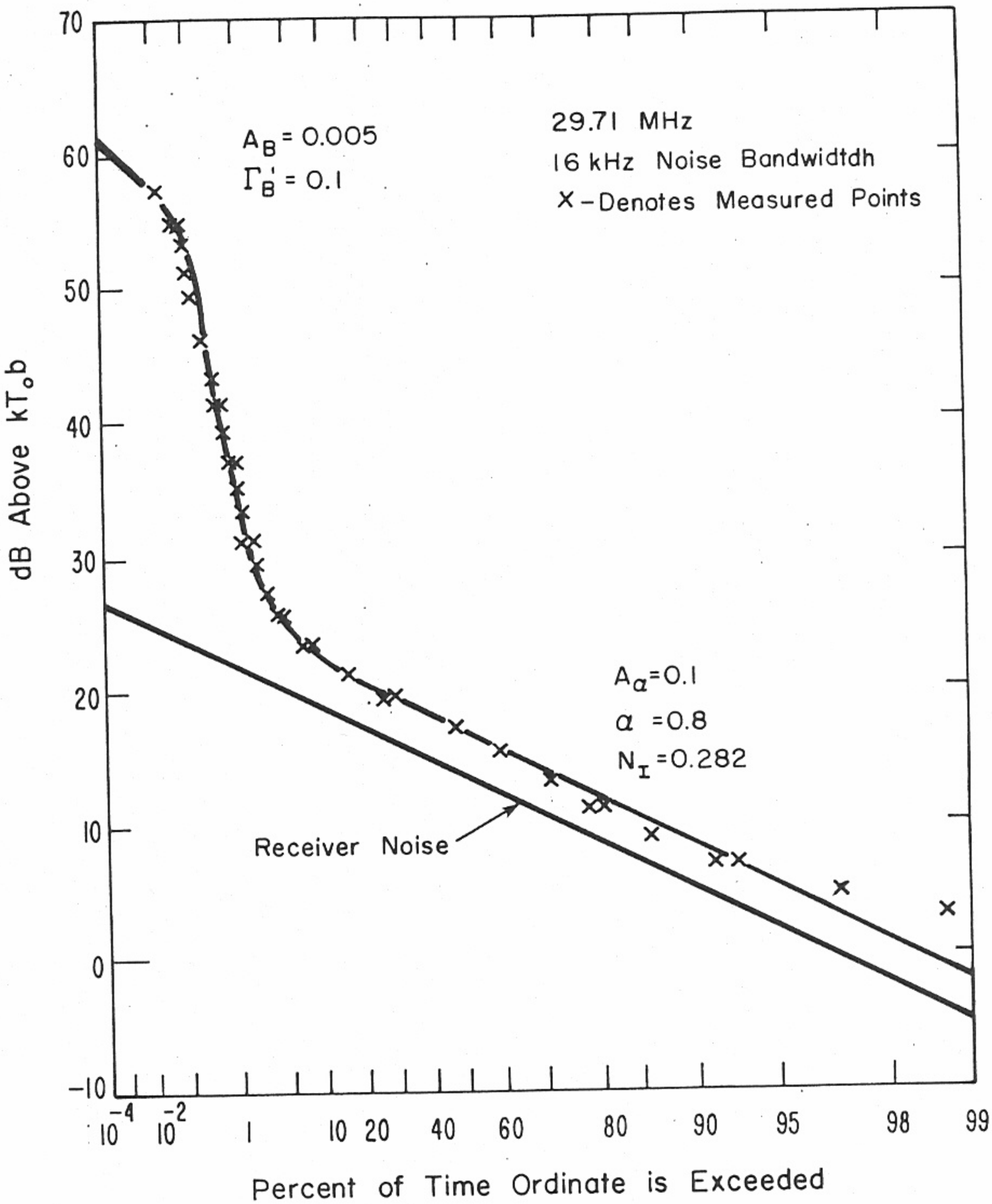

Figure 2.6. Comparison of measured envelope distribution $P_1(\mathcal{E} > \mathcal{E}_0)_B$ of automotive ignition noise from moving traffic with full Class B model, cf. (2.7a,b). [Data from Shepherd (1974), fig. 14.]

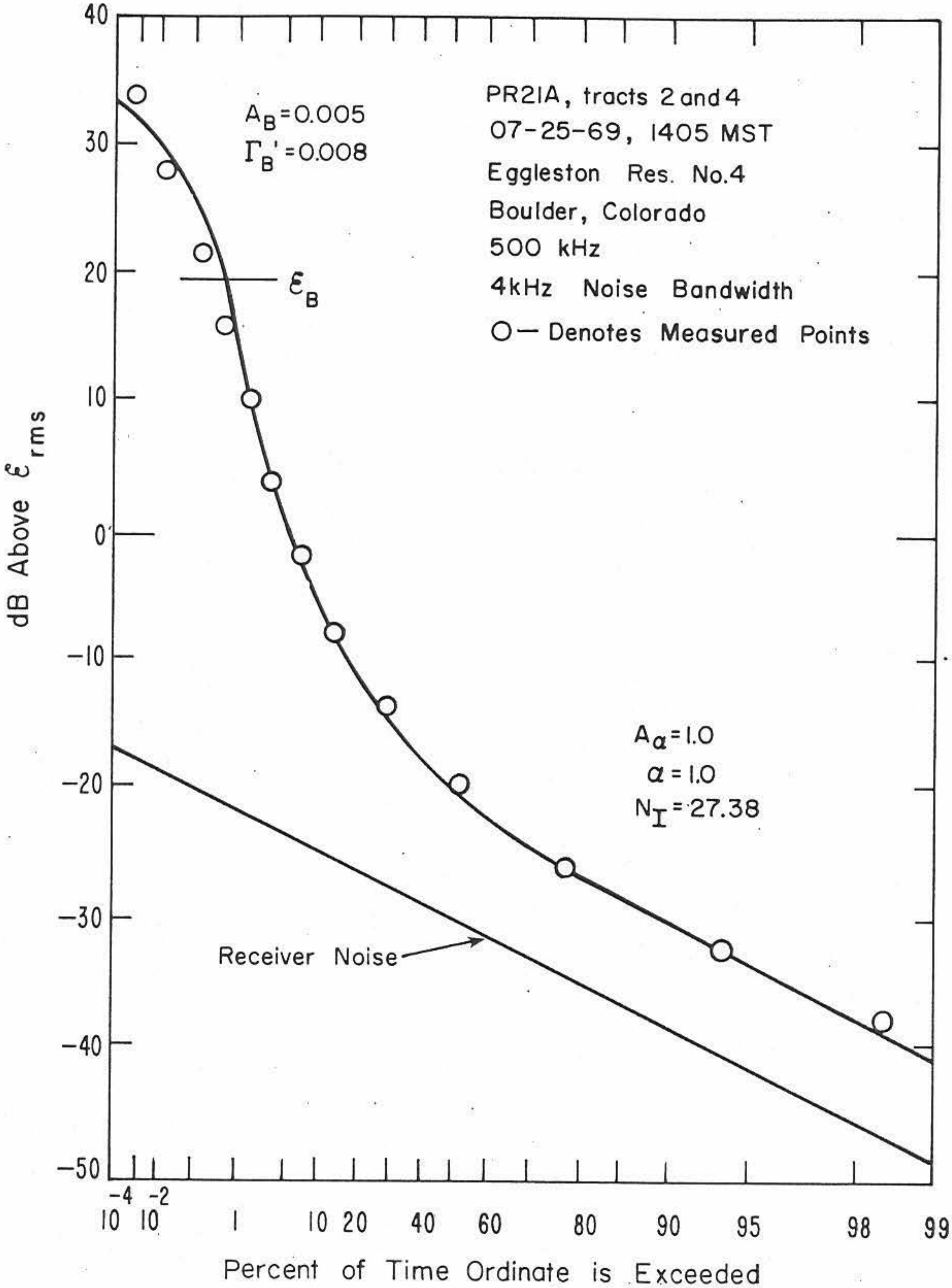

Figure 2.7. Comparison of measured envelope distribution $P_1(\mathcal{E} > \mathcal{E}_0)_B$ of atmospheric noise with full Class B model, cf. (2.7a,b). [Data from Esperland and Spaulding (1970), p. 42.]

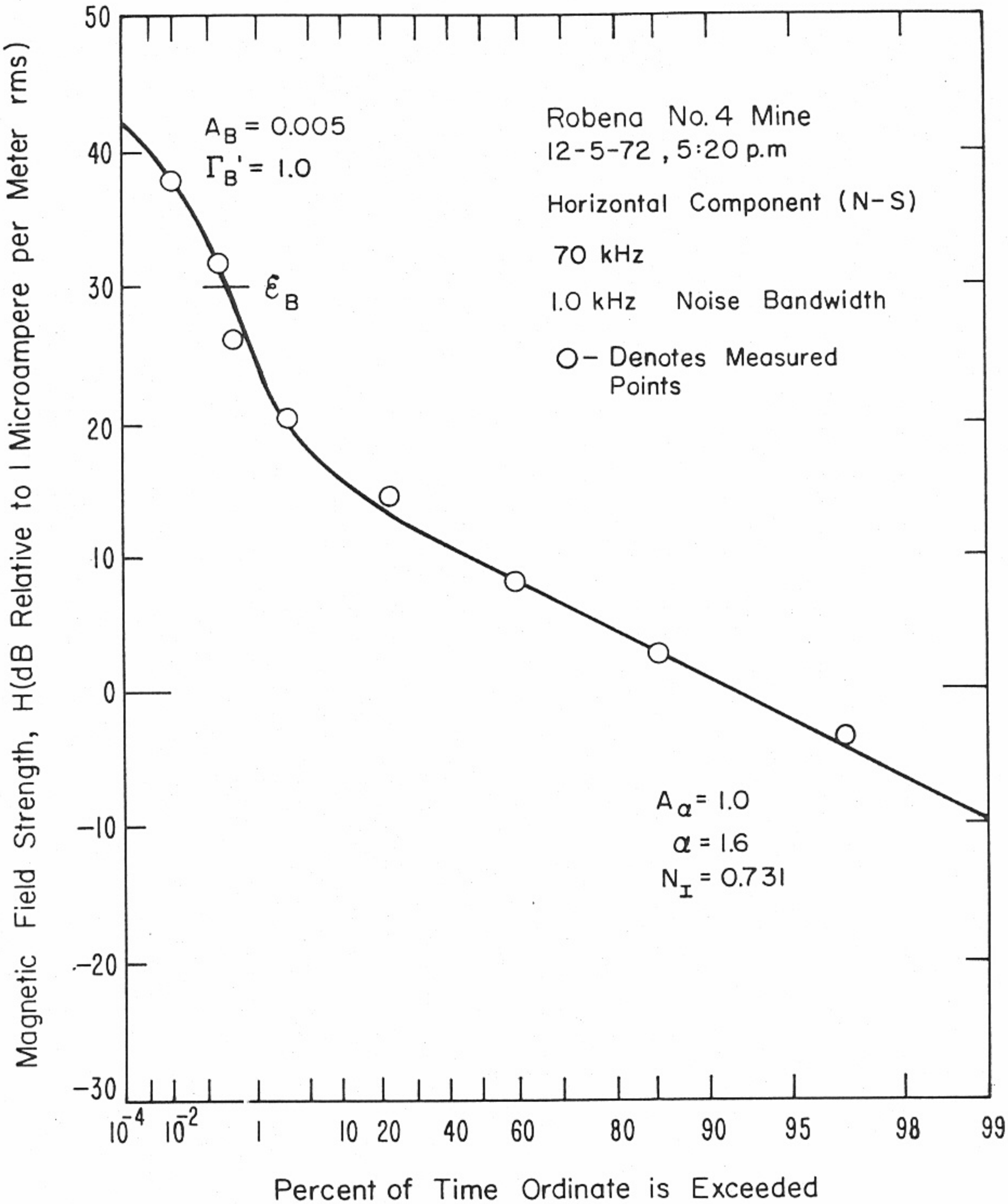

Figure 2.8. Comparison of measured envelope distribution $P_1(\mathcal{E} > \mathcal{E}_0)_B$ of man-made interference (mining machinery noise) with the full Class B model, cf. (2.7a,b). [Data from Bensema et al. (1974), fig. 67, p. 115.]

(4). The governing, physically structured parameters of these PD's and pdf's which are likewise also canonical, can be obtained from approximate experimental data (usually expressed as an APD).

The importance of the canonical character of these models cannot be overstressed: with such models we avoid the very limited and nonpredictive quality of all ad hoc models, whose structure must be verified and whose parameters provide little or no physical insight into the underlying process itself. Second, because these models are derived from physical principles, their parameters are physically defined, are consequently canonical, and are quantifiable in specific instances from empirical data. Their structure, however, is independent of any particular measurement.

Figures (2.1) and (2.2) show APD's, e.g. $P_1(\mathcal{E}>\mathcal{E}_0)_A$ vs. the normalized envelope threshold $\mathcal{E}_0$, for Class A interference, respectively from ore-crushing machining in a mine (data from Adams, Bensema, and Kanda [1974]), and and from a powerline (from E.C. Bolton, [1972]). Observe the characteristic very steep rise following the rayleigh region (constant slope), followed in turn by the expected bending over of the APD for the rarer "events", in each case. [Similar examples of Class A interference, but from man-made, intelligent sources, have also been observed; current experimental studies at ITS, Boulder, are underway to obtain additional such data.]

Figures 2.3-2.5 show APD's of Class B interference, respectively for (i), primarily urban automative ignition noise [Spaulding and Espeland, 1971]; (ii), atmospheric noise [Espeland and Spaulding, 1970]; (iii), flourescent lights, in a mine shop office [Adams et al, 1974]. Observe the more gradual departure from the straight-line rayleigh region, and the continuing rise, with constantly increasing slope in the Figures (which is equivalent to $\eta \to 0$ for $\exp(-a^2\mathcal{E}_0^\eta)$, as $\mathcal{E}_0 \to \infty$). [In these particular examples the inevitable "bend over" points $\mathcal{E}_B$, lie outside the range of data taken, e.g. for $P_{1-B}<10^{-6}$, so that we are able to obtain all the global parameters, except for A_B, cf. Sec. 6C, Part II.] This is not the case, however, for the Class B examples of Figs. (2.6)-(2.8), e.g., respectively for (i), ignition noise from vehicles moving on a freeway [Shephard, 1974]; (ii), atmospheric noise [Espeland and Spaulding, 1970];

and (iii), machinery noise in a coal mine [Bensema, Kanda, and Adams, 1974]. Here the required bend-over of the APD's is exhibited, along with the inflexion points, ε_B. In these cases we can obtain numerical estimates all the six global (and hence all the generic) parameters characteristic of each example of interference*, man-made or natural, by the methods briefly cited below in Section 2.5, and in more technical detail in Section 6, Part II.

Figures (2.1)-(2.8) are typical of Class A and Class B interference, man-made and natural. They are not intended to be exhaustive. Extensive additional APD data (mostly Class B) are available, for example in Espeland and Spaulding [1970], and Bensema, Kanda and Adams [1974], for example. [We have not included Class C APD data, although these appear in the references cited, because we limit our analysis and comparisons here to the essentially "pure" Class A and Class B interference environment, some (analytical) conditions for which are examined in Sec. 7, Part II.] Again, a striking feature of the present approach is its ability to handle an unlimited variety of noise sources, as long as the dominating Class is identified.

2.5 Remarks on the Estimation of Model Parameters:

We distinguish two sets of model parameters: (a), the so-called global parameters, which appear explicitly in the analytical forms for the APD's, etc. and (b) generic parameters, which are defined directly in terms of the underlying canonical, statistical-physical model. To some extent, both sets overlap. In any case, once the global parameters have been estimated from the data, which usually requires the calculation of the (first-order) APD, the generic parameters can be calculated from them.

The method for obtaining the global parameters is described in detail in Section 6, Part II, and will not be repeated here. However, for convenience, we list the two sets of parameters for each interference Class. These are [from Tables (6.1), (6.2), Part II]:

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*See footnote added p. 37.

$\underline{\text{Class A:}}$ $\underline{\text{Global:}}$ $(A_A, \Gamma'_A, \Omega_{2A})$; $\underline{\text{Generic:}}$ $(A_A, \sigma_G^2, \langle \hat{B}_{oA}^2 \rangle)$ (2.9)

$\underline{\text{Class B:}}$ $\underline{\text{Global:}}$ $(A_\alpha, \alpha, A_B, \Gamma'_B, \Omega_{2B}; N_I)$; $\underline{\text{Generic:}}$ $(A_B, \alpha, \sigma_G^2, \langle \hat{B}_{oB}^\alpha \rangle, \langle \hat{B}_{oB}^2 \rangle, N_I)$ (2.10)

where, in addition to those parameters described in Sections (2.2), (2.3) above, $\langle \hat{B}_{oA}^2 \rangle, \langle \hat{B}_{oB}^2 \rangle$ are the mean square envelopes of the basic waveforms emitted from the ARI receiver stage, cf. Eqs. (2.64a,b), Part II. Class A noise is described by a three-parameter model, while Class B interference is a six-parameter model globally, and similarly a five-parameter model generically, since the inflexion point ($\mathcal{E}_B$) is empirical and not deriveable here from the fundamental physical model itself. Again, we stress the fact that (except for $\mathcal{E}_B$) all model parameters are physically structured and hence are canonical <u>in form</u>; they are not ad hoc quantities, of strictly limited application.*

Finally, from an examination of the Class B model parameters vis-à-vis those of Class A, we note that (A_A, A_B), (Γ'_A, Γ'_B), and $(\Omega_{2A}, \Omega_{2B})$ are each identical types, of equivalent physical interpretation. The remaining Class B parameters are $(\alpha, \langle \hat{B}_{oB}^\alpha \rangle)$, <u>which provide additional information about the emitting sources</u>, e.g., source density, etc., basic waveshape. <u>Accordingly, it is suggested that to assess the interference environment more fully</u>, in addition to Class A measurements, when possible <u>ARI receiver bandwidths also be selected, to produce Class B interference at the output of the ARI-stage</u>, so as to obtain α and $\langle \hat{B}_{oB}^\alpha \rangle$, in addition to $(A_B, \sigma_G^2, \langle \hat{B}_{oB}^2 \rangle)$, which are analogous to the corresponding Class A set, cf. (2.9). [These $(A_B, \sigma_G^2, \langle \hat{B}_{oB}^2 \rangle)$ are, of course, modified from these Class A counterparts by this choice of ARI bandwidth.] In any case, the important new parameter, α, is obtained, which gives us an estimate of an effective mean source density with range, and the actual one (with range), if the governing propagation law (γ) is also known, or measured, cf. comments following (2.8b) above. Further information about source distributions may be obtained with the help of steerable, directional beam patterns, cf. Section (2.5), Part II. [We shall reserve these questions, in detail, to a succeeding study.]

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* N_I is not fully dependent on the other generic parameters and is independent of $\mathcal{E}_B$. Hence it may be regarded as generic, cf. Sec. 6B, 6C.

2.6 Some Additional Results:

A brief review of additional results obtained in this Report is now presented. We consider:

(1). First-Order Moments, $\langle \mathcal{E}^{\beta} \rangle$: These are obtained analytically for both Class A and B noise in Section 5 (Part II). They exist for all (real, finite) β, although the (approximate) expressions for the Class B cases are necessarily more complex then for Class A. Alternative, exact, closed-form relations are also obtained for the even integer moments (β=2,4,6,...), cf. Section 5.2 (Part II). [See also the discussion in Section 5.3 (Part II).]

(2). Conditions for Class A,B,C Noise: More precise, analytical conditions are derived in Section 7 (Part II) mutually to distinguish Class A, B, and C interference, than those qualitatively discussed in Section (1.1) above. In general, if the Impulsive Index of one component (A or B) greatly exceeds that of the other (B or A), then the former (A or B) dominates, and we have in practice Class A, or B noise. When this is not the case, the result is the the more general, Class C interference (which we shall treat in a subsequent study).

(3). Approach to Rayleigh Statistics: This occurs when either, or both, the Impulsive Index or the independent gaussian component becomes very large, cf. Section 2.4, Part II. (This is a consequence of the Central Limit Theorem in probability [cf. Section 7.7-3, Middleton, 1960].

(4). Hall Models: A primary empirical model, constructed earlier by Hall [1966], is frequently used for ad hoc representations of the interference environment. Our Class B results, upon deletion of the additive gaussian component (both from the impulsive and independent sources), can be shown to exhibit a Hall form, with Hall parameter (θ_{Hall}=2). [See Section 3.2B, Part II; also, Spaulding and Middleton [1975], Chapter 2.] Such models, however, have a variety of draw-backs, among them being their ad hoc character, with the parameter(s) entirely empirical, and the non-existence of the second moment, in many instances, as well as the non-existence of all moments $\langle \mathcal{E}^{\beta} \rangle$, where $\beta > \theta_{Hall} - 1$. Their principal advantage is analytic simplicity, which, however, does not ultimately compete with the physical-statistical models of the types developed here. These, though analytically much more involved, are nonetheless still tractable for the purposes of source and system analysis, cf. Spaulding and Middleton [1975]. No Hall models are deriveable from Class A models, however.

(5). Class A vs. Class B Interference: Some Summary Remarks: A concise comparison of some of the salient properties of Class A and Class B interference is presented in Table (2.1):

Table 2.1 Class A vs. Class B Interference

Class A	Class B
1. New Models and Results;	"Classical" (20 Yrs. Old), But New Approach; New Results
2. 3 Global and 3 Generic Parameters $(A_A, \Gamma'_A, \Omega_{2A}) : (A_A, \sigma_G^2, \langle \hat{B}_{oA}^2 \rangle)$	6 Global and Generic Parameters $(A_\alpha, \alpha, A_B, \Gamma'_B, \Omega_{2A}, N_I)$ $(A_B, \alpha, \sigma_G^2, \langle \hat{B}_{oB}^{\alpha} \rangle, \langle \hat{B}_{oB}^2 \rangle, N_I)$ $\hat{\mathcal{E}}_B$: empirical paramater of approximation
3. All Moments $\langle \varepsilon^{\beta} \rangle$, $0 \le \beta$ exist	All moments $\langle \varepsilon^{\beta} \rangle$, $0 \le \beta$ exist
4. Insensitive to Source Distribution in Space and Propagation Law; Canonical Forms;	Sensitive to Source Distribution and Propagation Law (α); Canonical Forms;
5. Waveform in IF Output: "Gaps" in Time $[P_1(\varepsilon=0)>0)$;	Waveform in IF Output: no "Gaps" in Time [e.g. $P(\varepsilon=0)=0$];
6. No Gaps in Time if Gaussian Background $\left\{\begin{matrix} X & \text{Gauss P.D.} \\ \mathcal{E} & \text{Rayleigh P.D.} \end{matrix}\right\}$ as $A_A \to \infty$; and/or $\sigma_G^2 \to \infty$	No Gaps in Time $(\sigma_G^2 \ge 0)$; $\left\{\begin{matrix} X & \text{Gauss P.D.} \\ \mathcal{E} & \text{Rayleigh P.D.} \end{matrix}\right\}$ as $(A_\alpha, A_B) \to \infty$; and/or $\sigma_G^2 \to \infty$
7. No Hall Models Exist;	Hall Models for Special Values of $\underline{\alpha}$; (Gauss Component Absent)

2.7 General Comments; Next Steps:

In the preceeding sections we have summarized the principal results of our present study of the (first-order) envelope and phase statistics of man-made and natural electromagnetic interference, whatever its physical origins and characteristics. These analytical models, of Class A and B interference, are mathematically tractible and canonical in application: the forms of the results, and the number, type, and general structure of the associated parameters, are invariant of the particular source. Of course, particular parameter values do depend on the specific properties of the particular source involved. These are estimated in turn, by general procedures outlined here., cf. Section 2.5 above, and Section 6, Part II from experimental data, principally the APD [= exceedance probability $P_1(\mathcal{E}>\mathcal{E}_0)$2 . The canonical character of these models and their parameters is derived from the general underlying physical structure upon which the models are based. This, in turn, is itself a general space-time model of propagation, source distribution, and emission [Middleton, 1974, and Section 2, and principally Secs. 2.1,2,5, Part II here].

As expected, the resulting statistics of amplitude and envelope are highly nongaussian (or nonrayleigh), as the analysis and examples in Part II and the experimental results of Section (2.4) indicate. This necessarily has a critical effect on conventional receiver and system operation, which may be in conventional usage, (approximately) optimized, e.g. "matched", to desired signals in gaussian noise (so-called correlation receivers and their extensions), but which is radically suboptimum for this kind of electromagnetic environment [Spaulding and Middleton, 1975].

Here we are concerned with first-order interference statistics themselves, not only for purposes of system design, optimization, and comparison, but also for the tasks of measuring and assessing the properties of EM interference fields. Excellent agreement between model and observation has been found, as the examples of Section 2.4 above demonstrate. In addition, explicit numerical results are also obtained for the global and generic parameters of the interference phenomenon in question, e.g., automotive ignition noise, communications, atmospherics, machinery, power line emissions and the like. These parameter values, along with the basic physical structure, permit us to deduce general properties of the interference field, such as average source distribution in space (α), emission density in time (Impulsive Index, A), mean intensity (Ω_2), the amount of external gaussian noise (σ_G^2), etc., and, of course, the associated APD, or exceedance probability $P_1(\mathcal{E}>\mathcal{E}_0)$, as well as various moments ($\langle \mathcal{E}^\beta \rangle$) of the interference process.

First-order statistics of these highly nongaussian EM noise environments as embodied in the APD, $P_1(\mathcal{E}>\mathcal{E}_0)$, $P_1(X>X_0)$, for example are, however, minimal for the proper treatment of the general class of communication systems operating in such environments. In many situations the performance bounds established from these first-order statistics are quite adequate [the independent sample cases of Spaulding and Middleton, 1975, for example], where higher-order time structures are not significant. However, when they are, one clearly needs appropriate extensions of the present models. In addition, the joint statistics of signals and noise are also required, of first- and higher-orders as well. Therefore, as part of our continuing effort to develop an applicable analytic description of the EM <u>nongaussian</u> interference environment, we present the following program, in approximate order of undertaking:

I. <u>Interference Models</u> (present series):

1). Report, Part III: First-order statistics of the instantaneous amplitude (X) for Class B noise, e.g., $P_1(X>X_0)_B$, $w_1(X>X_0)_B$, moments, parameter estimates, etc. (now underway).

2). Report, Part IV: First-order statistics of Class C noise, envelope (E) and instantaneous amplitude (X), e.g. $P_1(X>X_0)_C$, $P_1(\mathcal{E}>\mathcal{E}_0)_C$, etc., with experimental comparisons.

3). Report (possibly Part V) on measurements, parameter estimates, and description of EM interference environments. This will include evaluation of selected, earlier data [for example, Furutsu and Ishida, 1960, Espeland and Spaulding, 1970; Shephard, 1974], and comparisons with our models. This may also include recommendations for van usage and area coverage, etc.

4). Report, on simulation of EM environments, to establish robustness and sensitivity of the various models to modifications in their structure.

5). Report, on various special problems and extensions of Part I-IV, for example, to develop the analysis for Class B noise when $\alpha = 0, \alpha \geq 2$, including other source distributions. Also, we need to examine the case of a single source at known positions [Shephard, 1974]. The rôle of the "corrections" [Middleton, 1974] requires further clarification, etc.

6). Report on mean and variance of "zero-crossings" for Class A,B,C interference. These statistics are useful adjuncts to the APD's to provide some insight into the time-structure of the interference.

7). Report on the first-order statistics (envelope, phase, and instantaneous amplitude) of Class A and B interference with a general, additive signal present at the input to the typical narrowband receiver.

8). Report on the development of higher-order (principally 2nd-order) pdf's for Class A, B, C noise; possibly including signals as well.

II. <u>Performance and Optimum Systems in General EM Interference Environments:</u> (series with Spaulding and Middleton [1975]).

1). Report (Part II): Optimum reception with Class B interference; this is now underway);

2). Report (Part III): extension of the above to Class C cases;

3). Report, on the analysis of other specific systems in Class A and B noise; the improvement of performance bounds, the specification of LOBD receiver structures, and further extensions of the evaluation process.

These lists are not, of course, complete, nor will the program itself necessarily be carried out in the order indicated, since some of these topics may shift in priority as time goes on. In any case, however, the general goal of developing an applicable analytical theory, tested by experiment, for these general classes of man-made and natural, highly nongaussian, electromagnetic noise or interference processes, may be considered a major priority task for the future in the science and technology of telecommunications.

2.8 Acknowledgements

The author wishes to thank Dr. A. D. Spaulding of the Office of Telecommunications for his constructive discussions and criticisms of this material and for performing the numerical calculations and developing the figures contained in this report.

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*Note that $\Gamma_B' \equiv \sigma_G^2/\Omega_{2B}$, where σ_G^2 is the independent gaussian component, which is different from the total gauss component $\Delta\sigma_G^2 = \sigma_G^2 + b_{2\alpha}A_B$, cf., eq. (2.88a). Thus, in Figs. 2.6, 2.8 we must calculate Ω_{2B} from the data curve and then obtain σ_G^2 from Γ_B'. From the other parameters in these figures all the remaining generic parameters are then readily found. On the other hand, for Fig. 2.7 Ω_{2B} occurs at 0 dB, by normalization. Since P_1=0.36 determines the total gauss component ($\Delta\sigma_G^2$ for Class B, σ_G^2 for Class A noise), from the data of Fig. 2.7 we get $\Delta\sigma_G^2 \doteq -17$ dB ($=2.10^{-2}$) and $\therefore \sigma_G^2 = \Gamma_B'\Omega_{2B} = 10 \log_{10} (8.10^{-3}) \doteq -21$ dB, which gives in turn $b_{2\alpha}A_B \doteq 0.012$ (in units of Ω_{2B}). Again, all remaining generic parameters are now obtainable, from this and the other parameter data on Fig. 2.7 (also in units of Ω_{2B} here).

PART II. ANALYSIS

1. INTRODUCTION TO THE ANALYSIS

Part II of this Report is devoted to the analytical details of our statistical-physical interference models, where in particular we are concerned with the first-order probability density functions (pdf's) $w_1(E)$,$w_1(\mathcal{E})$ and exceedance probabilities (PD's) $P_1(E > E_0)$, $P_1(\mathcal{E} > \mathcal{E}_0)$ of the envelope of the input noise process following the combined (linear) aperture - RF - IF (or ARI) filtering stages of a typical narrow-band receiver. As we see below, three principal classes of interference process are defined: Class A noise, where (in qualitative language) the input noise is spectrally narrower than the ARI-filter at the receiver's front-end; Class B interference, where the reverse is true - this input process is spectrally broad vis-à-vis the ARI filter; and a general Class C noise, which consists of the sum of Class A and Class B components.

For Part II the material is organized as follows: In Section 2 below we develop the various forms of the first-order characteristic function (c.f.) for the envelope after reception in the ARI filter of the typical receiver. Included here are the Class A, B, and C noise types, with their associated descriptive parameters, and the modifications introduced by the geometrical effects of source distribution and propagation law. Section 3 is devoted to the determination of the envelope exceedance probabilities, for Class A and B interference, and Section 4 gives the associated pdf's. In Section 5 we determine the moments $\langle \mathcal{E}^\beta \rangle$, $0 \leq \beta$, and the conditions for their existence; Section 6 provides procedures for estimating the basic parameters of the various noise models from empirical data; and finally, in Section 7, we give quantitative, practical conditions for the applicability of our Class A or Class B models.

2. FORMULATION: THE CHARACTERISTIC FUNCTION

In this section we obtain the general forms of the first-order characteristic functions (c.f.'s) and probability densities (p.d.'s) and distributions (P.D.'s) for the "impulsive" interference of the various man-made and natural sources described in Part I above.

Our first step is to derive the desired general forms of the characteristic functions for the envelope of the received wave. The next step is to take advantage of the various physical conditions of the model, further to reduce our results to the particular expressions appropriate to the Class A and Class B interference, which can then be put in forms suitable for evaluation. A number of important parameters of these interference processes appear in the analysis and have important physical implications, which we shall develop further in the subsequent sections.

2.1 The Basic Statistical Model:

We assume as before [Middleton, 1974] for our basic model that there is an infinite number of potential sources in a source domain Λ, and that while the basic waveforms emitted all have the same form, their scale, durations, frequencies, etc., may be randomly distributed. Our fundamental postulate of this basic interference model is that: (i), the locations of the various possible emitting sources are poisson distributed in space; (ii), the emission times of the possible sources are similarly poisson distributed in time. Physically, this means that the sources are statistically independent, both in location and emission. Thus, by a slight generalization of earlier results [Middleton; 1967, 1972b, 1974], we can write for the first-order characteristic function of the instantaneous amplitude, X, of the received interference process

$$F_1(i\xi)_X = \exp\left\{\left\langle \int_{\Lambda,\hat{\varepsilon}} \rho(\underset{\sim}{\lambda},\hat{\varepsilon}) \left\langle e^{i\xi U(t;\underset{\sim}{\lambda},\underset{\sim}{\hat{\varepsilon}},\ldots)}-1 \right\rangle_{\underset{\sim}{\theta}} d\underset{\sim}{\lambda}\, d\hat{\varepsilon} \right\rangle\right\} . \qquad (2.1)$$

Here $\hat{\varepsilon}$ is an epoch, indicating vis-à-vis the receiver's (i.e. observer's) time t when a source may emit. The $\underset{\sim}{\lambda} = (\lambda,\theta,\phi)$ are coördinates, or a vector magnitude, appropriate to the geomtry of the source field, located in the region Λ, and of the receiver, with $d\underset{\sim}{\lambda}$ (= $d\lambda d\phi$) for a surface element; (= $d\lambda d\theta d\phi$) for a volume element. The quantity $\rho(\underset{\sim}{\lambda},\hat{\varepsilon})$ is the "process density" of this joint space-time poisson interference process, and is non-negative, and can be regarded as proportional to a probability density [cf. (2.28) below]. The $\langle\ \rangle_{\underset{\sim}{\theta}}$ denotes a statistical average, e.g. $\int_{[\underset{\sim}{\theta}]}[\ ]w_1(\underset{\sim}{\theta})d\underset{\sim}{\theta}$, over various random parameters ($\underset{\sim}{\theta}$) which may be pertinent to our source model, such as doppler, source amplitude and duration, etc.

The U are the typical waveforms of the emitting sources, <u>after reception by the</u> (assumed linear) <u>aperture - RF - IF stages of our "narrow-band" receiver</u>. The received process X is given by

$$X(t) = \int_{\underset{\sim}{Z}(=\underset{\sim}{\Lambda}\times\underset{\sim}{\theta})} U(t|\underset{\sim}{Z})dN(\underset{\sim}{Z}), \tag{2.2}$$

where the {dN} are a poisson point process (in $\underset{\sim}{\lambda}$ and $\hat{\varepsilon}$), such that

$$\langle X(t)\rangle = \int_{\underset{\sim}{Z}} U\langle dN\rangle = \left\langle\int_{\Lambda}\rho(\underset{\sim}{\lambda},\hat{\varepsilon})U(t;\underset{\sim}{\lambda},\hat{\varepsilon},\underset{\sim}{\theta})d\underset{\sim}{\lambda}\ d\hat{\varepsilon}\right\rangle_{\underset{\sim}{\theta}}, \tag{2.3a}$$

is the process mean (if any), and

$$\langle X(t_1)X(t_2)\rangle = \iint_{\underset{\sim}{Z}^2} U_1U_2\langle dN_1dN_2\rangle = \left\langle\int_{\Lambda}\rho(\underset{\sim}{\lambda},\hat{\varepsilon})U(t_1;\underset{\sim}{\lambda},\hat{\varepsilon},\underset{\sim}{\theta})U(t_2;\underset{\sim}{\lambda},\hat{\varepsilon},\underset{\sim}{\theta})d\underset{\sim}{\lambda}d\hat{\varepsilon}\right\rangle_{\underset{\sim}{\theta}}$$
$$+ \langle X(t_1)\rangle\langle X(t_2)\rangle \tag{2.3b}$$

is the general second-moment of X(t), under our basic poisson assumption above of source location and emission.* Higher moments may be similarly obtained.

Since we are interested here in the <u>envelope</u> of the received process X, which is always narrow-band, in as much as the receiver is itself narrow-band, we have to consider the new random variables X_c, X_s, representing the slowly-varying "in-phase" and "out-of-phase" components of X, viz.,

$$X(t) = X_c(t)\cos\omega_o t + X_s(t)\sin\omega_o t = \mathrm{Re}\left\{(X_c - iX_s)e^{i\omega_o t}\right\} = \mathrm{Re}\left\{\hat{X}_o e^{i\omega_o t}\right\} \tag{2.4}$$
$$= \mathrm{Re}\left\{E\ e^{i(\omega_o t-\psi)}\right\}, \tag{2.4a}$$

* For a general development of process statistics, not necessarily limited to the poisson case of independent sources, see, for example, recent work [Middleton,1974,1975b] in the development of generalized scattering models.

where now $\omega_0 (=2\pi f_0)$ is the central (angular) frequency of the final (≡IF) stage of the receiver, and

$$E = \sqrt{X_c^2+X_s^2} \quad ; \quad \psi = \tan^{-1}(X_s/X_c) \quad ; \quad \therefore \hat{X}_0 = X_c - iX_s = Ee^{-i\omega_0\psi} \tag{2.4b}$$

with

$$X_c = E \cos \psi, \quad X_s = E \sin \psi .$$

Here E, ψ are, respectively the envelope and phase of the narrow band received process X. In functional form, cf. (2.2), we can write alternatively

$$X(t) = \mathrm{Re}\left\{\int_{\underset{\sim}{Z}} [U_c(t|\underset{\sim}{Z}) - iU_s(t|\underset{\sim}{Z})]e^{i\omega_0 t} dN(\underset{\sim}{Z})\right\} \text{ , or} \tag{2.5a}$$

$$= \mathrm{Re}\left\{\int_{\underset{\sim}{Z}} e(t|\underset{\sim}{Z}) e^{i\omega_0 t - i\phi_s(t|\underset{\sim}{Z})} dN(\underset{\sim}{Z})\right\} , \tag{2.5b}$$

in terms of an envelope and phase, where

$$e(t|\underset{\sim}{Z}) \equiv \sqrt{U_c^2+U_s^2} \quad ; \quad \phi_s \equiv \tan^{-1}(U_s/U_c). \tag{2.5c}$$

Comparing (2.5) and (2.4) we see at once that

$$(X_c, X_s) = \int_{\underset{\sim}{Z}} (U_c, U_s) dN(\underset{\sim}{Z}) \quad ; \tag{2.6a}$$

$$E\, e^{-i\psi_s} = \int_{\underset{\sim}{Z}} e(t|\underset{\sim}{Z}) e^{-i\phi_s(t|\underset{\sim}{Z})} dN(\underset{\sim}{Z}) . \tag{2.6b}$$

The characteristic function which we need now is for the random variables X_c, X_s, namely,

$$F_1(i\xi, i\eta)_{X_c,X_s} = \exp\left\{\left\langle \int_{\Lambda} \rho(\underset{\sim}{\lambda}, \hat{\varepsilon})[e^{i\xi U_c + i\eta U_s} - 1]\right\rangle \underset{\sim}{d\lambda} d\hat{\varepsilon}\right\} , \tag{2.7}$$

which is the two-dimensional generalization of (2.1) required here. The corresponding p.d. is

$$W_1(X_c, X_s) = \iint_{-\infty}^{\infty} F_1(i\xi, i\eta)_{X_c,X_s}\, e^{-i\xi X_c - i\eta X_s} d\xi d\eta/(2\pi)^2 . \tag{2.8}$$

Also, we have, formally, the following expression for the joint first-order density of envelope E and phase ψ, in terms of the in-phase and out-of-phase components X_c, X_s of X.

$$W_1(E,\psi) = W_1(X_c,X_s)\left|\frac{\partial(X_c,X_s)}{\partial(E,\psi)}\right| = EW_1(E\cos\psi, E\sin\psi), \quad \left.\begin{array}{l} E \geq 0 \\ 0 \leq \psi < 2\pi \end{array}\right\}, \tag{2.9}$$

where W_1, (2.8), is now

$$W_1(E\cos\psi, E\sin\psi) = \iint_{-\infty}^{\infty} F_1(i\xi,i\eta)_{X_c,X_s}\, e^{-iE(\xi\cos\psi+\eta\sin\psi)}\,d\xi d\eta/(2\pi)^2, \tag{2.9a}$$

with F_1 therein given by (2.7)

To proceed further, we make use of a number of results from our earlier development of the physical model [Sec. 2.2, Middleton, 1974], to write for the narrow-band basic waveform U (at the output of the receiver's IF)

$$U = U_{nb} = B_o(t,\underset{\sim}{\lambda}|\hat{\varepsilon},\underset{\sim}{\theta})\cos\mu_d\Psi(t,\underset{\sim}{\lambda}|\hat{\varepsilon},\underset{\sim}{\theta}), \quad \mu_d = 1+\varepsilon_d, \tag{2.10}$$

where B_o (>0) is an envelope, whose detailed structure we shall consider in more detail later and where Ψ is a phase, which has the form

$$\Psi \equiv \omega_o(t-\lambda-\hat{\varepsilon}) - \mu_d^{-1}[\Phi_S(t-\lambda-\hat{\varepsilon},\underset{\sim}{\theta})+\phi_T(\underset{\sim}{\lambda},f_o)+\phi_R(\underset{\sim}{\lambda},f_o)] \;, \tag{2.10a}$$

in which Φ_S, ϕ_T, ϕ_R are respectively the typical source phase, and the phase angles of the source (T) and receiver (R), complex beam patterns [cf. Sec. 2.5 below]. [The quantity $\varepsilon_d(=\mu_d-1)$ is the sum of the relative dopplers between sources and the receiver, and is always small, $0(10^{-5,-6})$, in our applications, viz: $\varepsilon_d = 2v/c = 0(10^{-6})$ for $v = 0(10^5$ mph) so that the envelope B_o is independent of ε_d.

From the fact that $U_{nb} = U_c\cos\omega_o t + U_s\sin\omega_o t$, we see at once from (2.10) that

$$U_c = B_o\cos[\Phi_S'+\mu_d\omega_o(\lambda+\hat{\varepsilon})-\varepsilon_d\omega_o t];\; U_s = B_o\sin[\Phi_S'+\mu_d\omega_o(\lambda+\hat{\varepsilon})-\varepsilon_d\omega_o t], \tag{2.11}$$

where $\Phi_s' \equiv \Phi_s+\phi_T+\phi_R$. We now use polar coördinates

$$\xi = r\cos\phi \quad ; \quad \eta = r\sin\phi \ ; \quad |\partial(\xi,\eta)/\partial(r,\phi)| = r \tag{2.12a}$$

and transform from (ξ,η)-space to (r,ϕ)-space in (2.9a). Thus, we see that

$$d\xi d\eta \ F_1(i\xi,i\eta)_{X_c,X_s} = r\,\hat{F}_1(ir,\phi)drd\phi \ , \quad 0 < r < \infty; \quad 0 \leq \phi < 2\pi. \tag{2.12b}$$

The c.f. $\hat{F}_1$ is $F_1(ir\cos\phi, ir\sin\phi)_{X_c,X_s}$, (2.7), which with (2.11) now reduces explicitly to

$$\boxed{\hat{F}_1(ir,\phi) = \exp\left\{\left\langle\int_\Lambda \rho(\underset{\sim}{\lambda},\hat{\varepsilon})e^{irB_0\cos[\Phi_s'+\mu_d\omega_0(\lambda+\hat{\varepsilon})-\varepsilon_d\omega_0 t-\phi]}-1\right\rangle_{\underset{\sim}{\theta}} d\underset{\sim}{\lambda}d\hat{\varepsilon}\right\}.} \tag{2.13}$$

The first-order p.d. for the envelope and phase (E,ψ), (2.9), with the help of (2.12b), (2.9a), becomes

$$\boxed{w_1(E,\psi) = E\int_0^\infty rdr\int_0^{2\pi}\frac{d\phi}{(2\pi)^2}\,\hat{F}_1(ir,\phi)e^{-iEr\cos(\psi-\phi)}}$$

$$E > 0, \ 0 \leq \psi < 2\pi \ . \tag{2.14}$$

This is as far as we can go without further appeal to the physical model, in particular, to the statistics governing the locations $(\sim\underset{\sim}{\lambda})$ of the sources and the epoch $(\hat{\varepsilon})$ of interference emissions. We note, however, that the p.d. for the envelope alone is readily found, e.g.* the integration over ψ (in $0,2\pi$), is well-known [cf. (2.19) following)]:

$$\boxed{W_1(E) = \int_0^{2\pi} w_1(E,\psi)d\psi = E\int_0^\infty rJ_0(rE)dr\int_0^{2\pi}\hat{F}_1(ir,\phi)d\phi/2\pi.} \tag{2.15}$$

* As usual, functions of different arguments are different functions, e.g. $W_1(E)\neq W_1(\psi)\neq W_1(X_c,X_s)$, etc., unless it is otherwise stated.

In addition, we have respectively for the P.D., and exceedance probability, or APD (a posterior probability here, that E exceeds a level $E_0(>0)$) defined as usual by

$$D_1(E_0) \equiv \int_0^{E_0} W_1(E)dE \quad ; \; P_1(E \geq E_0) \equiv \int_{E_0}^{\infty} W_1(E)dE = 1-D(E_0), \tag{2.16}$$

the following results, where we have used

$$\int_0^z zJ_0(z)dz = zJ_1(z), \tag{2.16a}$$

viz:

$$\boxed{\begin{aligned} D_1(E_0) &= E_0\int_0^{\infty} J_1(rE_0)dr\int_0^{2\pi}\hat{F}_1(ir,\phi)d\phi/2\pi, \qquad E_0 \geq 0 \quad (2.17a) \\ P_1(E \geq E_0) &= 1-E_0\int_0^{\infty} J_1(rE_0)dr\int_0^{2\pi}\hat{F}_1(ir,\phi)d\phi/2\pi \; . \quad (2.17b) \end{aligned}}$$

Our results (2.13)-(2.17) are generalizations of earlier results [Furutsu and Ishida, 1960; Middleton, 1972b; Giordano, 1970], where our basic assumptions, so far, postulate only poisson distributions of source location and emissions, e.g. essentially independent sources. No restrictions on the specific character of the statistics of the source parameters are as yet introduced. It is for this reason that the characteristic function $\hat{F}_1$ depends on ϕ, as well as on r.

2.2 First Reduction of the c.f. $\hat{F}_1$: The Narrow-Band Receiver Condition

At this point we invoke certain properties of the basic waveform $B_0\cos[\phi_s'+\mu_d\omega_0(\lambda+\hat{\epsilon})-\omega_0\epsilon_d t-\phi]$ which appears in the exponent in the integrand of (2.13). We use the facts that (i), B_0, Φ_s, are both slowly-varying functions of λ; and (ii), the process density $\rho(\underset{\sim}{\lambda},\hat{\epsilon})$ is likewise slowly varying, vis-à-vis cos $\omega_0\mu_d\lambda$, sin $\omega_0\mu_d\lambda$. Employing the familiar expansion in Bessel functions, In addition, we have respectively for the P.D., and exceedance probability, or APD (a posterior probability here, that E exceeds a level $E_0(>0)$) defined as usual by

$$D_1(E_0) \equiv \int_0^{E_0} W_1(E)dE \quad ; \; P_1(E \geq E_0) \equiv \int_{E_0}^{\infty} W_1(E)dE = 1-D(E_0), \tag{2.16}$$

the following results, where we have used

$$\int_0^z zJ_0(z)dz = zJ_1(z), \tag{2.16a}$$

viz:

$$\boxed{D_1(E_0) = E_0\int_0^{\infty} J_1(rE_0)dr\int_0^{2\pi}\hat{F}_1(ir,\phi)d\phi/2\pi, \qquad E_0 \geq 0} \tag{2.17a}$$

$$\boxed{P_1(E \geq E_0) = 1-E_0\int_0^{\infty} J_1(rE_0)dr\int_0^{2\pi}\hat{F}_1(ir,\phi)d\phi/2\pi \ .} \tag{2.17b}$$

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$$\exp[ia\cos\Phi] = \sum_{m=0}^{\infty} i^m \epsilon_m J_m(a)\cos m\Phi\ , \tag{2.18}$$

in (2.13), we see that only for m=0 does the integrand (containing the exponent) contribute, as all the other terms are highly oscillatory in regions where B_0, Φ_s, and ρ are slowly changing with λ. The result is the important simplification of (2.13) to

$$\hat{F}_1(ir,\phi) = \exp\left\{\left\langle \int_{\Lambda} \rho(\underset{\sim}{\lambda},\hat{\varepsilon})[J_0(rB_0[t,\underset{\sim}{\lambda}|\hat{\varepsilon},\underset{\sim}{\theta}] -1]\right\rangle_{\underset{\sim}{\theta}} d\underset{\sim}{\lambda}d\hat{\varepsilon}\right\} = \hat{F}_1(ir), \tag{2.19}$$

which is valid, <u>provided that U_{nb} is truly narrow-band</u>, e.g. $\Delta f_{ARI} << f_0$: the (composite) bandwidth of the (linear) aperture-RF-IF receiver stages is much less than the (IF) central frequency f_0. [The result (2.19) is now in the same <u>form</u> as obtained in earlier work, but still somewhat more general in detail.]

<u>The important analytic feature of (2.19) is that</u> now, because of the narrow-band receiver condition $\Delta f_{ARI} << f_0$, <u>the c.f. $\hat{F}_1$ is independent of ϕ.</u> Accordingly, we see that (2.14), (2.15), (2.17) reduce at once to the simpler forms (with the help of (2.18):

$$\boxed{w_1(E,\psi) = \frac{E}{2\pi}\int_0^{\infty} rJ_0(rE)\hat{F}_1(ir)dr = W_1(E)W_1(\psi),} \tag{2.20}$$

with

$$\boxed{W_1(E) = E\int_0^{\infty} rJ_0(rE)\hat{F}_1(ir)dr\ ;\ W_1(\psi) = 1/2\pi\ ,\ 0 \le \phi < 2\pi,} \tag{2.21}$$

and

$$D(E_0) = E_0\int_0^{\infty} J_1(rE_0)\hat{F}_1(ir)dr\ , \tag{2.22a}$$

$$P(E>E_0) = 1 - E_0\int_0^{\infty} J_1(rE_0)\hat{F}_1(ir)dr\ . \tag{2.22b}$$

Respectively for the PD and exceedance probability, P, for these narrow-band interference waves (in the n.b. receiver), the first-order p.d. of phase ψ is seen to be uniform over an (IF) cycle ($T_0 = 1/f_0$). The results (2.20)-(2.22b) are formally identical to those derived by Furutsu and Ishida [1960, Eqs. (2.9)-(2.11)], and by Giordano [1970], and, Giordano and Haber [1972], for example. Furthermore, $\hat{F}_1(ir)$ is clearly a Hankel transform of $W_1(E)$, from (2.21) and the fact that the inverse of (2.21) is

$$\hat{F}_1(ir) = \int_0^\infty J_0(rE)W_1(E)dE \equiv \left\langle J_0(rE)\right\rangle_E \quad . \tag{2.23}$$

[This relation is easily established with the help of

$$\sqrt{ab}\int_0^\infty J_m(ax)J_m(bx)xdx = \delta(b-a), \tag{2.24}$$

cf. p. 943, Morse and Feshbach [1953], applied to $\langle J_0(rE)\rangle_E$, with (2.21) for $W_1(E)$.] An equivalent expression, now in terms of the average over E and ψ, is obtained at once from the fact that

$$J_0(rE) = \int_0^{2\pi} e^{irE\cos\psi}\frac{d\psi}{2\pi} = \int_0^{2\pi} e^{irE\cos\psi}W_1(\psi)d\psi = \left\langle e^{irE\cos\psi}\right\rangle_\psi , \tag{2.24a}$$

[cf. (2.2.)], and from the relation (2.23), viz:

$$\boxed{\hat{F}_1(ir) = \left\langle e^{irE\cos\psi}\right\rangle_{E,\psi} ,} \tag{2.24b}$$

from which is seen the fact that $\hat{F}_1$ here is also the joint c.f. of envelope and phase, as expected. We shall use this result later in the calculation of moments, (cf. Sec. 5.2).

Next, let us look at the process density $\rho(\underset{\sim}{\lambda},\hat{\varepsilon})$: we write

$$(2.25)\quad \begin{cases} \rho(\underset{\sim}{\lambda},\hat{\varepsilon}) = \rho_\Lambda(\underset{\sim}{\lambda}|\hat{\varepsilon})\nu_T(\hat{\varepsilon}): & \text{(av. no. of emitting sources (in } \Lambda\text{) per unit domain) and emitting at } \hat{\varepsilon}\text{, in the interval } d\hat{\varepsilon}\text{) x (av. no. of emissions, per source, per interval } d\hat{\varepsilon}\text{) in the observation interval } T; \\ \quad = \rho_\Lambda(\underset{\sim}{\lambda})\nu_T(\hat{\varepsilon}): & \text{av. no. of emissions per unit domain } (d\Lambda) \text{ and per interval } d\hat{\varepsilon} \text{ in } T. \text{ This last is on the reasonable assumption that location and emission are independent "events".} \end{cases}$$

Then, we observe further that

$$\rho_\Lambda(\underset{\sim}{\lambda}) = \sigma_\Lambda(\underset{\sim}{\lambda})|J_\Lambda(\underset{\sim}{\lambda})| \ ; \quad \left.\begin{aligned} |J_\Lambda| &= c^2\lambda \text{ for surfaces} \\ &= c^3\lambda^2\sin\theta, \text{ for volumes,} \end{aligned}\right\} \text{[Sec. 2.3, Middleton (1974)]} \ , \tag{2.26}$$

in which σ_Λ is the physical density of emitting sources in the source domain Λ, We now define

$$\begin{cases} A_\Lambda \equiv \int_\Lambda \rho_\Lambda(\underset{\sim}{\lambda})d\underset{\sim}{\lambda}: & \text{av. no. of emitting sources in } \Lambda; \qquad (2.27a) \\ A_{\varepsilon,T} \equiv \int_T \nu_T(\hat{\varepsilon})d\hat{\varepsilon}: & \text{av. no. of emissions (per source) in the observation period, } T. \qquad (2.27b) \\ A_T \equiv \int_{\Lambda,T} \rho(\underset{\sim}{\lambda},\hat{\varepsilon})d\underset{\sim}{\lambda}d\hat{\varepsilon} = A_\Lambda A_{\varepsilon,T}: & \text{av. no. of emissions in the period } T. \qquad (2.27c) \end{cases}$$

Consequently, we can also define probability densities for source location and emission by

$$w_1(\underset{\sim}{\lambda}) \equiv \rho_\Lambda(\underset{\sim}{\lambda})/A_\Lambda \ ; \quad w_1(\hat{\varepsilon})_T \equiv \nu_T(\hat{\varepsilon})/A_{\varepsilon,T}. \tag{2.28}$$

Now let us look at the integrand, $\hat{I}$, of (2.19), and use (2.25)-(2.28) to write

$$\hat{I}_T = A_\Lambda A_{\varepsilon,T}\left\langle \int_{\Lambda,\hat{\varepsilon}} w_1(\underset{\sim}{\lambda}) w_1(\hat{\varepsilon}) J_0(rB_0[t-\lambda-\hat{\varepsilon};\underset{\sim}{\lambda},\underset{\sim}{\theta}])-1 \; d\underset{\sim}{\lambda} d\hat{\varepsilon} \right\rangle , \tag{2.29}$$

where explicitly we have from earlier work [Middleton, 1972b, 1974] for the received envelope B_0 of a typical emission

$$B_0 = |\mathcal{A}_R(\underset{\sim}{\lambda},f_0)\mathcal{A}_T(\underset{\sim}{\lambda},f_0)|A_{oT}(t-\lambda-\hat{\varepsilon}|\underset{\sim}{\theta})g(\underset{\sim}{\lambda}), \tag{2.30}$$

where

$\mathcal{A}_R, \mathcal{A}_T$ = (complex) beam patterns of receiver and typical interfering source;
A_{oT} = (real) envelope of the source emission;
$g(\underset{\sim}{\lambda})$ = a geometric factor, which describes the propagation law, from source to receiver (which are assumed to be in each other's far field).

(2.30a)

[For this receiver, although the aperture may be comparatively broad-band, as may be that of the source, it is the narrowest filter, of the combination (aperture x RF x IF) which is controlling. By assumption, one or more of these filters is very narrow vis-à-vis f_0, cf. the comments following (2.19), so that the effective aperture response here is determined essentially by the response at (and about) f_0, cf. (2.30).]

Next we let

T_S = duration of a typical emission, <u>at the IF output</u> (which may be $\lessgtr T$); (2.31a)

this can generally be a random variable (one of the $\underset{\sim}{\theta}$ in (2.29)). We now let

$$t-\lambda-\hat{\varepsilon} \equiv y = \bar{T}_s z \; , \tag{2.31b}$$

where $\bar{T}_s$ is the mean duration of an emission (at the IF output), and z is dimensionless, to rewrite (2.29) as

$$\hat{I}_T = A_\Lambda A_{\varepsilon,T}\bar{T}_s\left\langle\int_0^{T_s/\bar{T}_s} w_1(t-\lambda-\bar{T}_s z)_{\hat{\varepsilon}}\{J_0[r\hat{B}_0(z,\underset{\sim}{\lambda};\underset{\sim}{\theta}')]-1\}\right\rangle_{T_s,\underset{\sim}{\lambda},\underset{\sim}{\theta}} dz \quad (2.32)$$

all $(t \in T)$.

Here $\underset{\sim}{\theta}'$ are any other random parameters in $\hat{B}_0$, e.g. amplitude of the basic emission envelope B_0 (in the receiver, at output of IF); $\hat{B}_0$ itself is $B_0(z\bar{T}_s,\underset{\sim}{\lambda};\underset{\sim}{\theta}')$. Further reduction is obtained by writing

$$A_{0T}(\bar{T}_s z|\underset{\sim}{\theta}) = A_0 e_{0\gamma} u_0(z) \ , \quad (u_0(z) = 0,\ z > T_s/\bar{T}_s \ ,\ z < 0) \qquad (2.33)$$

where

$$\begin{cases} A_0 = \text{(peak) amplitude of the received envelope (at output of the IF);} \\ e_{0\gamma} = \text{a limiting "voltage" setting (in suitable dimensions), at which} \\ \quad \text{the receiver will respond to a test signal, above the receiver} \\ \quad \text{noise,* at output of the IF;} \\ u_0(t) = \text{normalized envelope wave form at output of receiver IF.} \end{cases}$$

(2.34)

Note from (2.33) that the generic waveform $u_0(z)$ is, of course, required to vanish outside the time interval during which the typical emission ($\sim A_0 e_{0\gamma} u_0$) is "on", e.g. for $\bar{T}_s z > T_s$, $\bar{T}_s z < 0$.

Finally, let us multiply and divide $\hat{I}$ by T, the observation period, and write (2.32) as

$$\hat{I}_T = \left(\frac{A_\Lambda A_{\varepsilon,T}}{T}\right)\bar{T}_s\left\langle\int_0^{T_s/\bar{T}_s}\{Tw_1(t-\lambda-z\bar{T}_s)_{\hat{\varepsilon}}\}[J_0(r\hat{B}_0(z,\underset{\sim}{\lambda}|\underset{\sim}{\theta}'))-1]\right\rangle_{\underset{\sim}{\theta}',\underset{\sim}{\lambda},T_s} dz,$$

(2.34)

all $(t \in T)$. Also, we see that

---

* The precise definition of $e_{0\gamma}$ can be determined by standard decision-theoretical techniques of detection [Middleton, 1960; Chapters 18, 19], with some appropriate choice of false alarm rate.

$$\therefore \begin{cases} A_\Lambda A_{\epsilon,T}/T \equiv \bar{\nu}_T: & \text{av. no. of emissions per second, in the observation period } T, \quad (2.35a) \\ \nu_T \bar{T}_s \equiv \gamma_T: & \text{"density" of the process [cf. Sec. 11.2; Eq. (11.74), Middleton, 1960]: (av. no. of emissions per second)x(mean duration of an emission).} \quad (2.35b) \end{cases}$$

Equation (2.34) is a generalization of earlier results, which permits the treatment of nonstationary régimes.

At this point we restrict our attention to the most common situation of "local stationarity", whereby it is assumed that there are no changes in average source numbers and emission properties during the observation period T, and that the emission probability $w_1(\hat{\epsilon})$ is uniform, e.g. $Tw_{1-\hat{\epsilon}} = 1$, for all allowed values of z. Thus, (2.34) reduces to the basic form

$$\text{[uniform p.d. of } \hat{\epsilon}\text{]:} \quad \boxed{\hat{I}_T(r) = \bar{\nu}_T \bar{T}_s \left\langle \int_0^{T_s/\bar{T}_s} [J_0(r\hat{B}_0)-1]dz \right\rangle_{\underset{\sim}{\lambda}, T_s, \underset{\sim}{\theta}'}} , \quad (2.36)$$

which is the one from which we develop our subsequent analysis (beginning with (2.38)) in this Report. Furthermore, for the idealized steady-state situation where $T \to \infty$, we write

$$\left.\begin{aligned} &\lim_{T\to\infty} \bar{\nu}_T = \bar{\nu}_\infty \quad \left(= A_\Lambda \lim_{T\to\infty} (A_{\epsilon,T}/T)\right) \\ &\lim_{T\to\infty} \gamma_T = \gamma_\infty \end{aligned}\right\} \quad (2.37)$$

and, accordingly, (2.36) becomes

[uniform p.d. of $\hat{\epsilon}$]:

$$\boxed{\hat{I}_\infty(r) = A_\infty \left\langle \int_0^{T_s/\bar{T}_s} [J_0(r\hat{B}_0)-1]dz \right\rangle_{\underset{\sim}{\lambda}, T_s, \underset{\sim}{\theta}'}} \; ; \; \boxed{A_\infty \equiv \gamma_\infty = \bar{\nu}_\infty \bar{T}_s} \; : \quad (2.38)$$

This limiting form of (2.36) is the expression which we shall exploit in the remainder of the study.

The quantity A_∞ appearing in (2.38) is

$$A_\infty(=\gamma_\infty): \ \underline{\text{impulsive index}} \text{ (of the present analysis)*} \qquad (2.39)$$

As we have already noted in our earlier studies [Middleton, 1972b,1973,1974], the Impulsive Index is a measure of the temporal "overlap" or "density", at any instant, of the superposed interference waveforms at the receiver's IF output. It is one of the key parameters of the interference model, in that it critically influences the character of the p.d.'s and P.D.'s of the interference, as observed at the output of the initial (linear) stages of a typical narrow-band receiver. With small values of A_∞ the statistics of the resultant output waveform are dominated by the overlapping of comparatively few, deterministic waveforms, of different levels and shapes, so that the interference has an "impulsive", somewhat structured appearance. For increasingly large values of A_∞ the resultant approaches a normal, or gaussian process, as one would expect from the Central Limit Theorem [Middleton, 1960, Sec. 7.7], as we shall see in more detail later [cf. Sec. 2.4].

2.3 Interference Classes A, B, and C: The Rôle of Input and Receiver Bandwidths:

We are now ready to examine the basic form, (2.38), of $\hat{I}_\infty(r)$ [$=\log \hat{F}_1(ir)$]. The rôle of the duration T_s of a typical emission (as perceived at the output of the ARI (≡ aperture x RF x IF) stages of the narrow-band receiver) is critical in determining the form of $\hat{I}_\infty(r)$.

Let us consider first the important special case when the emission duration T_s is fixed. From Eqs. (2.63a,b), (2.70), (2.72a) of Middleton [1960] we may write for the envelope $\hat{B}_0$, cf. (2.10), (2.30), (2.31a), (2.33),

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* $A_T(\equiv A_\Lambda A_{\varepsilon,T}=\bar{\nu}_T T$, cf. (2.27c),(2.35)) was designated "impulsive index", A, in the author's earlier treatments [Middleton, 1972b,1973,1974].

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$$
\left.\begin{aligned}
\hat{B}_0(z;\underset{\sim}{\lambda},\underset{\sim}{\theta}') &= \left|\int_{-\infty}^{\infty} E_0(\tau;\underset{\sim}{\lambda},\underset{\sim}{\theta}')e^{-i\Phi_0(\tau;\underset{\sim}{\lambda},\underset{\sim}{\theta}')} h_0(z\bar{T}_s-\tau|\underset{\sim}{\lambda})_{ARI}\right. \\
&\qquad \left.\cdot e^{-i\gamma_0(z\bar{T}_s-\tau)+i\omega_D\tau}\, d\tau\right| \\
&= \left|\int_{-\infty}^{\infty} S_{in}(f';\underset{\sim}{\lambda},\underset{\sim}{\theta}')_0 Y_0(i\omega';\underset{\sim}{\lambda})_{ARI}\, e^{i\omega' z\bar{T}_s}\, df'\right| \\
&= A_0 e_{0\gamma}|\mathcal{Q}_T(\underset{\sim}{\lambda};f_0')|g(\underset{\sim}{\lambda})u_0(z)\ .
\end{aligned}\right\} \qquad (2.40)
$$

Here h_0, γ_0 are real, and $h_{ARI}(t) = 2h_0(t)_{ARI} \cos[\omega_0 t-\gamma_0(t)]$ is the weighting function of the composite ARI filter. The (narrow-band) system function Y_0 is obtained from the fourier transform $Y_{0\text{-}ARI} = \underset{\sim}{\mathcal{F}}\{h_0 e^{-i\gamma_0}\}$ and ω_D $[=\omega_c-\omega_0]$ measures the amount of "detuning" of the input signal (at ω_c, shifted to the IF region) from the (trial) central frequency (f_0) of the ARI stage.

With T_s fixed, we have in general the situation shown in Fig. (2.1)II for the envelope of the narrow-band output of the ARI filter, produced by a typical interference emission of finite duration, T_{in}. The output envelope ($\sim u_0(z)$) produced by a typical input interference envelope [shown as a rectangular pulse in Fig. (2.1)II], always consists of two parts: a part which we shall call Class A with normalized envelope $u_{0A}(z)$, which is produced by the input emission ($\sim E_{0\text{-}in}$), which is "on" during the interval $0 \leq z \leq T_{in}$ $(= T_{sA} = \bar{T}_{sA})$; and a part we shall term Class B, with normalized envelope $u_{0B}(z-1)$ $(\neq u_{0A}(z))$, which represents the transient decay of the output of the ARI filter, following the termination of the input emission $[\sim E_0(z)_{in}]$. The sum of Class A and Class B envelopes is called <u>Class C</u>, e.g. $u_{0C}(z) = u_{0A}(z) + u_{0B}(z-1)$, [cf. Fig. (2.1)II, where, of course, $u_{0B}=0$, $z < 1$, $u_{0A} = 0$; $z < 0$, $z > 1$ in our definition. Thus, all receiver outputs are typically Class C, with variable amounts of Class A and Class B, depending on the duration of the typical input interference waveform vis-à-vis the response time of the ARI filter at the front-end of our receiver. Equivalently, the relative extents of the Class A and B components depend generally on the ratio of the bandwidth of the input (Δf_{in}) to the

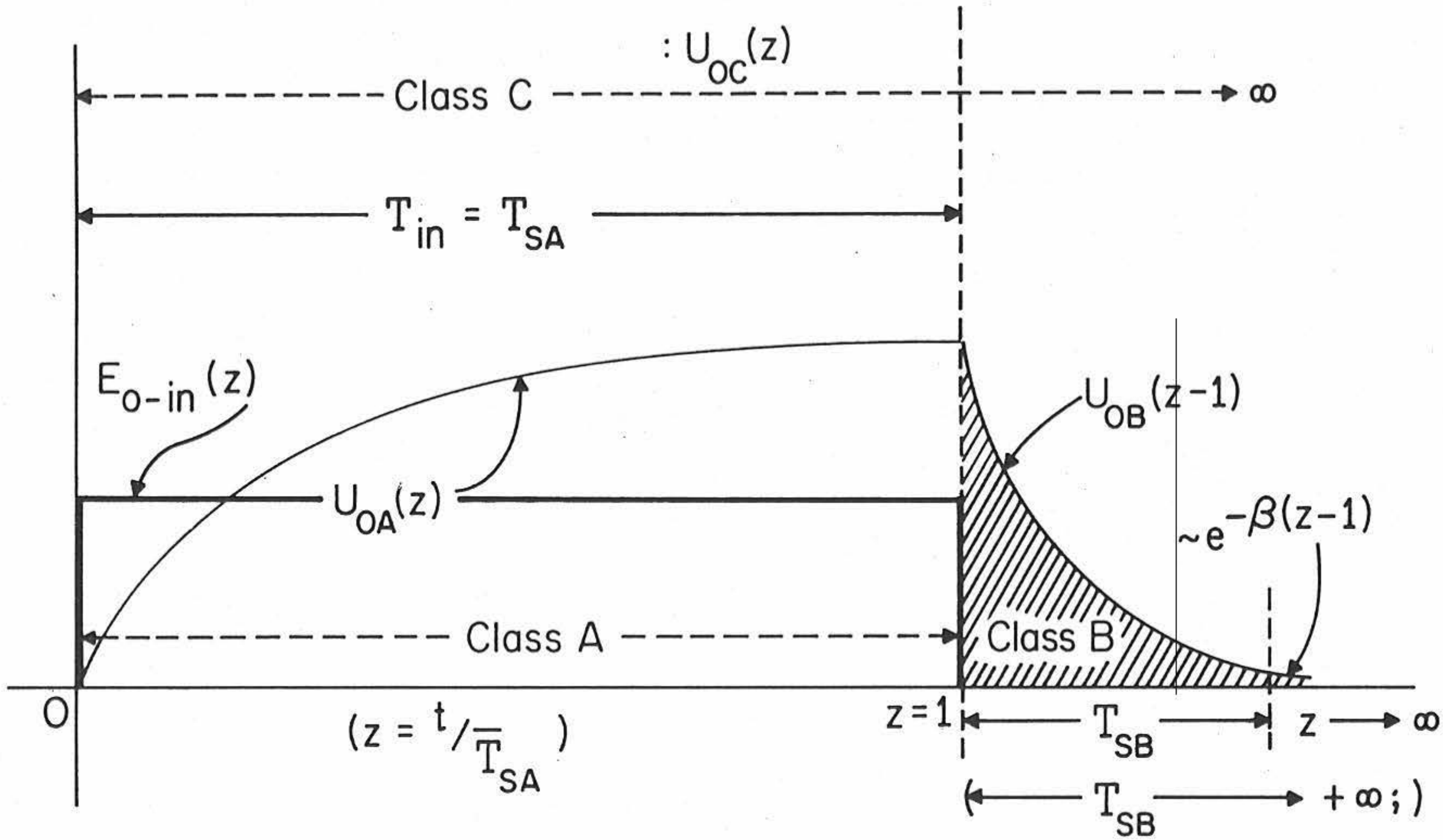

Figure 2.1 (II). Typical output envelope of ARI filter output in (narrowband) receiver, showing Class A and Class B envelope components (positive halves only); Class C = Class A + Class B.

bandwidth Δf_{ARI} of the linear "front-end" of the receiver. With $\Delta f_{in} \gg \Delta f_{ARI}$, for example, the usual case of very wide band interference (automobile ignition, fluorescent lights, atmospheric noise), T_{sA} is very small vis-à-vis $\bar{T}_{sB}$: the build-up time ($=T_{in}$) is very brief for Class A, while the duration of Class B depends on the decay time ($\sim \Delta f^{-1}_{ARI}$) of the ARI stage, which is much longer than T_{in}. On the other hand, with narrowband inputs of long duration [$\Delta f_{ARI} \gg \Delta f_{in}$], the transient at the termination of the typical input is of negligible effect vis-à-vis the Class A component. For comparable bandwidths ($\Delta f_{ARI} \sim \Delta f_{in}$) both Class A and Class B make comparable contributions, e.g. neither can be ignored vis-à-vis the other, so that we have then generally the Class C waveform in the receiver. [In all cases $\int_0^\infty u_{oA,B}(t)^k dt$, $k > 0$, are finite.]

From (2.38) we see accordingly that $\hat{I}_\infty(r)$ can now be written as the sum of the Class A and Class B components, viz: ($T_{in} = \bar{T}_{sA}$ fixed for the moment):

$$\boxed{\begin{aligned}\hat{I}_\infty(r) &= \nu_\infty\left\{\bar{T}_{sA}\int_0^{z=1}\left\langle J_o(r\hat{B}_{oA})-1\right\rangle_{\underset{\sim}{\theta}',\underset{\sim}{\lambda}} dz + \bar{T}_{sB}\int_0^\infty\left\langle J_o(r\hat{B}_{oB})-1\right\rangle_{\underset{\sim}{\theta}',\underset{\sim}{\lambda}} dz\right\} \\ &= \hat{I}_\infty(r)_A + \hat{I}_\infty(r)_B \equiv \hat{I}_\infty(r)_C ,\end{aligned}}$$

(2.41)
(2.41a)

on changing variable $z' \to z$ in $u_{oB}(z'-1) \to u_{oB}(z)$, e.g.

$$\hat{B}_{oA,B} = A_o e_{o\gamma}^{(A,B)}|\mathcal{A}_{RT}(\underset{\sim}{\lambda},f_o')|g(\underset{\sim}{\lambda})(u_{oA}(z),u_{oB}(z)), \tag{2.41b}$$

from (2.40). In terms of the characteristic function (2.19) we see at once that

$$\hat{F}_1(ir) = \hat{F}_1(ir)_A \cdot \hat{F}_1(ir)_B \equiv \hat{F}_1(ir)_C \tag{2.42}$$

with the important result that Class C interference consists of the independent sum of Class A and Class B components, as defined above. Note, also, that the limiting voltages $e_{o\gamma}^{(A)} \neq e_{o\gamma}^{(B)}$, generally, as the receiver responds to "narrow-band" inputs (A) differently from "broad-band" (B).

Specifically, when we can ignore the Class B component [$\bar{T}_{sB}(<\infty) << \bar{T}_{sA}$, e.g. sufficiently narrow-band input vis-à-vis the receiver*], we have here, from (2.42), (2.41), in (2.19),

$$\boxed{\begin{aligned} \hat{F}_1(ir)_C \to \hat{F}_1(ir)_A &= \exp\left\{A_{\infty,A}\int_0^1 \left\langle J_0(r\hat{B}_{oA})-1\right\rangle_{\underset{\sim}{\theta}',\underset{\sim}{\lambda}} dz\right\} \qquad (2.43a)\\ &= e^{-A_{\infty,A}}\cdot\exp\left\{A_{\infty,A}\left\langle J_0(r\hat{B}_{oA})\right\rangle_{z,\underset{\sim}{\lambda},\underset{\sim}{\theta}'}\right\}, \qquad (2.43b)\end{aligned}}$$

$$(\bar{\nu}_\infty\bar{T}_{sA}=)\ A_{\infty,A} >> A_{\infty,B}(=\bar{\nu}_\infty\bar{T}_{sB}), \tag{2.43b}$$

where the averages $\langle\ \rangle_{z,\underset{\sim}{\lambda},\underset{\sim}{\theta}'}$ are explicitly

$$\left\langle\ \right\rangle_{z,\underset{\sim}{\lambda},\underset{\sim}{\theta}'} = \int_0^1 dz \int_{\Lambda,\underset{\sim}{\theta}'} \frac{w_1(\underset{\sim}{\theta}')\rho(\underset{\sim}{\lambda})}{A_\Lambda}[\ ]\, d\underset{\sim}{\lambda}d\underset{\sim}{\theta}' . \tag{2.43c}$$

Similarly, when the Class A component is ignorable [$\bar{T}_{sA} = T_{sin} << T_{sB}$ e.g. very broad-band inputs vis-à-vis the receiver's ARI stages*] we get

$$\boxed{\hat{F}_1(ir)_C \to \hat{F}_1(ir)_B = \exp\left\{A_{\infty,B}\int_0^\infty \left\langle J_0(r\hat{B}_{oB})-1\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'} dz\right\},} \qquad (2.44)\quad A_{\infty,B}>>A_{\infty,A},$$

with the averages given by (2.43c), (without the average over z). Note that when $r \to \infty$, $\hat{F}_{1-A} \to \exp(-A_{\infty,A})$, while $\hat{F}_{1-B,C} \to 0$. This means, as we shall see in detail in Section 3.1 later, that for Class A interference

* The precise conditions for effectively Class A or Class B interference alone are developed in Sec. 7 later.

there will be a non-zero probability of "gaps-in-time", i.e. finite (nonzero) intervals in the receiver's output when there is no waveform present, while for Class B and C interference there is always a nonvanishing waveform level and hence no "gaps-in-time". [Of course, physically there is always some inherent system noise, which makes it strictly impossible to have a true "gaps-in-time" situation.]

We remark, again, that Class A [and consequently Class C] interference models are new. The earlier "classical" analyses [Rice (1945); Middleton (1953); Furutsu and Ishida (1960); Giordano (1970; Giordano and Haber (1972)], for example, all dealt with Class B interference, and for the most part in much less general terms and by different modes of approximation.*

2.3.1 Some Extensions:

Usually, there is an accompanying gaussian background noise, which may arise in a number of ways:

(i). as system noise in the receiver;
(ii). as external interference, which is the resultant of many independent sources, none of which is exceptionally dominant with respect to the others (so that the Central Limit Theorem applies);
(iii). as a mixture of receiver noise and (independent) gaussian external interference. (2.45)

From (2.12b) and the gaussian counterpart of (2.7), viz.

$$\hat{F}_1(i\xi,i\eta)_{X_c,X_s:\text{gauss}} = e^{-(\xi^2+\eta^2)\sigma_G^2/2} \quad , \tag{2.46}$$

we readily find that

$$\hat{F}_1(ir,\phi)_{\text{gauss}} = e^{-\sigma_G^2 r^2/2} = \hat{F}_1(ir)_G; \quad \sigma_G^2 = \sigma_R^2+\sigma_E^2 \quad , \tag{2.47}$$

* Technically, Giordano [1970] and Haber [1972] express their results in a Class A format, whenever sample size (T) is finite, cf. remarks in Section (5.3) following.

cf. (2.13), where σ_R^2, σ_E^2 are respectively the receiver and external noise variances.

Applying (2.47) to (2.15), (2.17) shows directly that*

$$W_1(E)_G = E\int_0^\infty rJ_0(rE)e^{-r^2\sigma_G^2/2}\,dr = \frac{Ee^{-E^2/2\sigma_G^2}}{\sigma_G^2} \quad , \quad E \geq 0 \tag{2.48a}$$

so that

$$D_1(E_0)_G = 1-e^{-E_0^2/2\sigma_G^2} \quad ; \quad P_1(E \geq E_0) = e^{-E_0^2/2\sigma_G^2} \quad , \quad (E_0 \geq 0) \quad . \tag{2.48b}$$

As expected, the p.d. and P.D. here are rayleigh.

Our results of Sec. 2.3 above are readily extended <u>to include the more general situation of interference inputs of random duration</u>, e.g. $T_{in} = T_{sA}$ ($\neq \bar{T}_{sA}$) generally. Only the Class A portion of $\hat{I}_\infty(r)$, (2.41), is modified. Letting $z_0 \equiv T_{in}/\bar{T}_{sA}$, we have at once, for the desired extension of (2.43),

$$\hat{F}_1(ir)_A = e^{-A_{\infty,A}} \exp\left\{\left\langle\int_0^{z_0} J_0(r\hat{B}_{0A})\,dz\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'}\right\} \quad . \tag{2.49}$$

Combining (2.49) and (2.44) with (2.47) gives us the desired characteristic functions with which we shall be concerned here, and subsequently, in this report:

<u>Class A Interference and Gauss:</u>

$$\hat{F}_1(ir)_{A+G} = e^{-\sigma_G^2 r^2/2-A_{\infty,A}} \exp\left\{A_{\infty,A}\left\langle\int_0^{z_0} J_0(r\hat{B}_{0A})\,dZ\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'}\right\} \tag{2.50}$$

<u>Class B Interference and Gauss:</u>

$$\hat{F}_1(ir)_{B+G} = e^{-\sigma_G^2 r^2/2} \exp\left\{A_{\infty,B}\int_0^\infty \left\langle[J_0(r\hat{B}_{0B})-1]\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'}\,dz\right\} \quad . \tag{2.51}$$

---

* Use Eq. (A.1-49) [Middleton (1960)], for example.

[We shall reserve the analysis for Class C interference and gauss noise, e.g. based here on

$$\hat{F}_1(ir)_{C+G} = \exp\left[-\sigma_G^2 r^2/2 + A_{\infty,A}\left\langle\int_0^{z_0}[J_o(r\hat{B}_{oA})-1]dz\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'}\right.$$

$$\left. + A_{\infty,B}\int_0^\infty\left\langle[J_o(r\hat{B}_{oB})-1]\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'}dz\right] \quad , \tag{2.52}$$

to a subsequent Report.]

2.4 Large Impulsive Indexes:

When the impulsive index, A_∞, is large, we expect asymptotically gaussian statistics for the instantaneous amplitude X [Secs.3,p.26; 5,p. 39, Middleton, 1974], and rayleigh statistics here, cf. (2.48), for the instantaneous envelope E. This latter is easily shown by developing $\hat{I}_\infty(r)$, (2.41) or (2.52), as a power series in r about r = 0, in the usual way.* Thus, the c.f. (2.52) for our general Class C case, with gaussian background noise in addition, becomes

$$\hat{F}_1(ir)_{C+G} = \exp\left\{-\frac{r^2}{2}\sigma_0^2\right\}\cdot\exp\left\{\sum_{n=2}^{\infty}\frac{(-1)^n r^{2n}}{2^{2n}(n!)^2}\left(A_{\infty,A}b_{2n}^{(A)} + A_{\infty,B}b_{2n}^{(B)}\right)\right\}, \tag{2.53}$$

where

$$\sigma_0^2 = (\sigma_R^2+\sigma_E^2) + \bar{\nu}_\infty\bar{T}_{sA}\left\langle\int_0^{z_0}\hat{B}_{oA}^2 dz\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'} + \bar{\nu}_\infty\bar{T}_{sB}\int_0^\infty\left\langle\hat{B}_{oB}^2\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'}dz, \tag{2.53a}$$

$$b_{2n}^{(A)} = \left\langle\int_0^{z_0}\hat{B}_{oA}^{2n}dz\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'} \quad , \quad b_{2n}^{(B)} = \int_0^\infty\left\langle\hat{B}_{oB}^{2n}\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'}dz \quad . \tag{2.53b, 2.53c}$$

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* Provided we consider for the moment finite observation intervals $T(<\infty)$, i.e. finite upper limits on the z-integrals in (2.51), (2.52), so that these integrals are uniformly convergent, proper integrals, permitting a series expansion of their integrands. Then, we ultimately have $\hat{F}_1(ir)_{C+G}: \lim_{T\to\infty} F_1(ir|T)_{C+G}$, where $(\lim_{T\to\infty})$ is invoked for each term of the resulting expansions. See the comments in B, Sec. (5.2) below.

[We shall reserve the analysis for Class C interference and gauss noise, e.g. based here on

$$\hat{F}_1(ir)_{C+G} = \exp\left[-\sigma_G^2 r^2/2 + A_{\infty,A}\left\langle \int_0^{z_0} [J_o(r\hat{B}_{oA})-1]dz \right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'} + A_{\infty,B}\int_0^{\infty}\left\langle [J_o(r\hat{B}_{oB})-1]\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'} dz\right] \quad , \tag{2.52}$$

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where

$$\sigma_0^2 = (\sigma_R^2+\sigma_E^2) + \bar{\nu}_\infty \bar{T}_{sA}\left\langle\int_0^{z_0}\hat{B}_{oA}^2 dz\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'} + \bar{\nu}_\infty\bar{T}_{sB}\int_0^\infty\left\langle\hat{B}_{oB}^2\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'} dz, \tag{2.53a}$$

$$b_{2n}^{(A)} = \left\langle\int_0^{z_0}\hat{B}_{oA}^{2n}dz\right\rangle_{z_0,\underset{\sim}{\lambda},\underset{\sim}{\theta}'} \quad , \quad b_{2n}^{(B)} = \int_0^\infty\left\langle\hat{B}_{oB}^{2n}\right\rangle_{\underset{\sim}{\lambda},\underset{\sim}{\theta}'} dz \quad . \tag{2.53b, 2.53c}$$

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Next, expanding the second exponential gives us

$$\hat{F}_1(ir)_{C+G} \doteq e^{-r^2\sigma_0^2/2}\left\{1+\frac{r^4}{2^4(2!)^2}B_4+\left[-\frac{r^6}{2^6(3!)^2}B_6+\frac{r^8}{2^8(4!)^2}18B_4^2\right]+\dots\right\}, \tag{2.54}$$

where now we write

$$B_{2n} \equiv A_{\infty,A}b_{2n}^{(A)} = A_{\infty,B}b_{2n}^{(B)}, \tag{2.54a}$$

cf. (2.53b,c).

From the leading term of (2.54) applied to (2.21), (2.22) we see that the result is indeed the expected rayleigh form, cf. (2.48a,b). Using the Hankel exponential integral relation, for example (A.1-49), [Middleton (1960)], viz.

$$\int_0^\infty J_\nu(az)z^{\mu-1}e^{-b^2z^2}dz = \frac{\Gamma(\frac{\nu+\mu}{2})}{2b^\mu\Gamma(\gamma+1)}\left(\frac{a}{2b}\right)^\nu {}_1F_1\left(\frac{\nu+\mu}{2};\nu+1;-\frac{a^2}{4b^2}\right), \tag{2.55}$$

$$\mathrm{Re}(\mu+\nu) > 0; \quad |\arg b| < \pi/4,$$

we obtain the Edgeworth "correction" terms to $W_1(E)_{C+G}$, $P_1(E > E_0)_{C+G}$, e.g. terms $O(A_\infty^{-1}, A_\infty^{-2}$, etc.) vis-à-vis the leading, rayleigh term. The first-order p.d. W_1 of the envelope and the APD, P_1, with correction terms, are found to be explicitly:

$$\boxed{\begin{aligned} W_1(E)_{C+G} \simeq \frac{Ee^{-E^2 2\sigma_0^2}}{\sigma_0^2}\Bigg\{1 &+ \frac{(B_4/\sigma_0^4)}{2^2 2!}\,{}_1F_1(-2;1;E_0^2/2\sigma_0^2) \\ &+ \left[-\frac{(B_6/\sigma_0^6)}{2^3 3!}\,{}_1F_1(-3;1;E_0^2/2\sigma_0^2)\right. \\ &+ \left.\left[\frac{18B_4^2}{2^4 4!\sigma_0^8}\right]{}_1F_1(-4;1;E_0^2/2\sigma_0^2)\right] + \dots\Bigg\}, \end{aligned}} \tag{2.56}$$

where

$$ {}_1F_1(-2;1;x^2) = 1-2x^2+x^4/4; \; {}_1F_1(-3;1;x^2) = 1-3x^2 + \frac{3}{2}x^4 - \frac{x^6}{6}; $$

$$ {}_1F_1(-4;1;x^2) = 1-4x^2 + 6x^4 - \frac{2}{3}x^6 + \frac{x^8}{24}, \text{ etc.}, \tag{2.56a} $$

with $x^2 = E^2/2\sigma_0^2$ here.

Similarly, we get for $P_1(E > E_o)_{C+G}$, (2.22b), here

$$ \boxed{\begin{aligned} P_1(E>E_o)_{C+G} \cong 1 - \left(\frac{E_o^2}{2\sigma_o^2}\right) e^{-E_o^2/2\sigma_o^2} \Bigg\{ & {}_1F_1(1;2;E_o^2/2\sigma_o^2) \\ & + \frac{(B_4/\sigma_o^4)}{2^2 3!} \, {}_1F_1(-1;2;E_o^2/2\sigma_o^2) \\ & + \left[-\frac{(B_6/\sigma_o^6)}{2^3 3!} \, {}_1F_1(-2;2;E_o^2/2\sigma_o^2) \right. \\ & \left. + \frac{(18B_4^2)\sigma_o^{-8}}{2^4 4!} \, {}_1F_1(-3;2;E_o^2/2\sigma_o^2) \right] + \dots \Bigg\}, \end{aligned}} \tag{2.57} $$

with

$$ \begin{aligned} {}_1F_1(1;2;x^2) &= (e^{x^2}-1)/x^2 \; ; \quad \text{[Eq. (A.1.19b), Middleton (1960)]} \\ {}_1F_1(-1;2;x^2) &= 1 - x^2/2 \; ; \\ {}_1F_1(-2;2;x^2) &= 1 - x^2 + x^4/6 \; ; \\ {}_1F_1(-3;2;x^2) &= 1 - 3x^2/2 + x^4/2 - x^6/24, \text{ etc.} \end{aligned} \tag{2.57a} $$

The first two terms of (2.57) reduce to

$$ P_1(E>E_o)_{C+G} \simeq e^{-E_o^2/2\sigma_o^2} \quad , \tag{2.57b} $$

as expected, for this cumulative rayleigh P.D., cf. (2.48b).

Finally, the above results apply also for the purely Class A or Class B interference, whenever $\bar{\nu}_\infty$ becomes very large (i.e. the impulsive index is large). The variance σ_0^2, (2.53a), is then suitably modified, as is (2.54a) for the correction terms. Since B_{2n} is $O(A_\infty)$, while σ_0^2 is also $O(A_\infty)$, it is clear that the correction coefficients $B_4(\sigma_0^4$, $[B_6/\sigma_0^6$, $(B_4^2)\sigma_0^{-8}]$ are $O(A_\infty^{-1}, A_\infty^{-2})$, etc., showing the rate of fall-off of the correction terms with increasing index A_∞.

2.5 Second Reduction of the c.f. $\hat{F}_1$: The Rôle of Source Distribution and Propagation Law:

Our major problem, as stated earlier in Part I, is to obtain analytically tractable results, as well as a pertinent physical foundation for our models of man-made (and natural) electromagnetic interference. Technically, our central problem now is to evaluate the probability densities and cumulative probabilities (2.21), (2.22), when the interference is Class A or Class B, accompanied by gaussian noise, with the respective characteristic functions (2.50), (2.51). [The detailed study of Class C interference, with the more general c.f. (2.52), is reserved to Part IV of this series of Reports.]

The desired evaluation may now be achieved by recalling (as in Section 3 of Part I [Middleton, 1974]) that the general character of the p.d. (and hence of the P.D.) of a random variable at large values is controlled principally by the behavior of the associated characteristic function at and near zero values of its argument. Thus, the behaviour of $\hat{F}_1(ir)$ at, and in the vicinity of, $r=0$, is determined by the largest r-dependent contribution which establishes the large-amplitude structure of $W_1(E)$, $P_1(E)$, etc., i.e. as $E \to \infty$. In fact, for these general classes of non-gaussian noise this corresponds to the expected slower fall-off of $W_1(E)$, as $E \to \infty$, than the rayleigh p.d. (2.48a), for example here. [See also the discussion in Section (2.7)A following.]

Our preliminary procedure for obtaining the required development of the c.f. $\hat{F}_1$ in the neighborhood of $r=0$ consists of: (i), expressing J_0-1 as an integral; (ii), using an explicit class of propagation law and source distribution; (iii), reversing the order of integration in (i), (ii) as expected, for this cumulative rayleigh P.D., cf. (2.48b).

Finally, the above results apply also for the purely Class A or Class B interference, whenever $\bar{\nu}_\infty$ becomes very large (i.e. the impulsive index is large). The variance σ_0^2, (2.53a), is then suitably modified, as is (2.54a) for the correction terms. Since B_{2n} is $O(A_\infty)$, while σ_0^2 is also $O(A_\infty)$, it is clear that the correction coefficients $B_4(\sigma_0^4$, $[B_6/\sigma_0^6$, $(B_4^2)\sigma_0^{-8}]$ are $O(A_\infty^{-1}, A_\infty^{-2})$, etc., showing the rate of fall-off of the correction terms with increasing index A_∞.

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Our preliminary procedure for obtaining the required development of the c.f. $\hat{F}_1$ in the neighborhood of $r=0$ consists of: (i), expressing J_0-1 as an integral; (ii), using an explicit class of propagation law and source distribution; (iii), reversing the order of integration in (i), (ii) and observing the bounds imposed by the fact that u_{oA} is of finite duration, while $u_{oB}(z) \neq 0$, $0<z<\infty$, cf. Fig. (2.1) above.* As we shall see below, it is this latter condition (on u_{oA} vs. u_{oB}) which critically affects the explicit form of the needed development of F_1.

We begin with the identity $-J_o'(x) = J_1(x)$, from which it follows by integration that

$$J_o(y) - 1 = -\int_0^y J_1(x)dx. \tag{2.58}$$

Then, the exponents of the c.f.'s $\hat{F}_{1A,B}$ are (without the contributions, for the moment, of the background normal noise) but with the help of (2.58),

$$\boxed{-A_{\infty,A,B}\left\langle\int_0^{(z_o,\text{or}\infty)} dz\int_0^{x=r\hat{B}_{oA,B}} J_1(x)dx\right\rangle_{z_o,\underset{\sim}{\lambda},\underset{\sim}{\theta}'} = \hat{I}_{\infty}(r)_{A,B}.} \tag{2.59}$$

2.5.1 Propagation Law:

We now introduce the somewhat restrictive condition that the source distribution and propagation law are expressible in the factored form: $a(\lambda)[b(\phi)$ or $b(\theta,\phi)]$. The beam patterns are always independent of distance ($c\lambda$), e.g.

$$|\mathcal{A}_{RT}(\underset{\sim}{\lambda};f_0')| = |\mathcal{A}_{RT}(\phi;f_o')|_{\text{plane}};\quad |\mathcal{A}_{RT}(\theta,\phi;f_o')|_{\text{volume}}, \tag{2.60a}$$

e.g.

$$|\mathcal{A}_{RT}(\underset{\sim}{\lambda};f_o')| = |\mathcal{A}_{RT}[(\hat{\underset{\sim}{i}}_T,\hat{\underset{\sim}{i}}_R)f_o'/c]|\ ,\ (\hat{\underset{\sim}{i}}_R=-\hat{\underset{\sim}{i}}_T,\text{cf. Fig. (2.1), Middleton [1974]}), \tag{2.60b}$$

where specifically

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* Our procedure here is a generalization of that used by Giordano [1970], who, however, considered what in the limit ($T\to\infty$) is ultimately only Class B interference, and only special choices of source distributions and propagation law.

$$\hat{\underset{\sim}{i}}_R = \hat{\underset{\sim}{i}}_x \cos\phi_R \sin\theta_R + \hat{\underset{\sim}{i}}_y \sin\phi_R \cos\theta_R - \hat{\underset{\sim}{i}}_z \cos\theta_R \tag{2.60c}$$

in which ϕ_T is an azimuthal angle and θ_T a polar angle, as sketched in Fig.(2.2)II. Thus, for the propagation law, $g(\underset{\sim}{\lambda})$ in (2.41b) we write

$$g(\underset{\sim}{\lambda}) = [g_S(\phi),\ g_V(\theta,\phi)]/(4\pi c\lambda)^{\gamma}\ , \quad 0 < \gamma \tag{2.61}$$

where $g_{S,V}$ are angular factors, usually taken to be unity in the common propagation models. In general, $\gamma > 0$, and, in fact, $\gamma \geq 1/2$: $\gamma = 1/2$ corresponds to the "wave guide" modes often encountered in long-distance propagation in the atmosphere, while $\gamma = 1$ applies for the usual spherical spreading of less distant sources. For practical applications, sources and receiver in a common plane, Fig.(2a)II, is typical of most mobile land transport communication environments, while the "volume" situation of Fig. (2b)II is characteristic of ground/air, or ground/satellite, or air/air environments. Also, for practical purposes, atmospheric noise may often be regarded as essentially coplanar with the surface (and $\gamma \doteq 1/2$), unless the principal discrete sources are comparatively near to the receiver, i.e. $\theta_R(=\theta_T)$ is large, e.g. [>0(5-10°)].

2.5.2 Source Distributions:

For the moment we continue to assume that the source distributions are factorable into the form $\sigma \equiv a(\lambda)b(\theta,\phi)$, cf. remarks at the beginning of Sec. (2.5.1) above. Then, the density $w_1(\lambda)$ required in the averages $\langle\ \rangle_\lambda$ in (2.59) is now from (2.26), (2.28)

$$w_1(\underset{\sim}{\lambda}) = \begin{bmatrix} \sigma_S(\lambda)c^2\lambda\sigma_S(\phi) \\ \\ \sigma_V(\lambda)c^3\lambda^2\sin\theta\ \sigma_V(\theta,\phi) \end{bmatrix} A_{S,V}^{-1}\quad , \tag{2.62}$$

for the surface and volume régimes, where the normalizing factors $A_{S,V}$ are given by

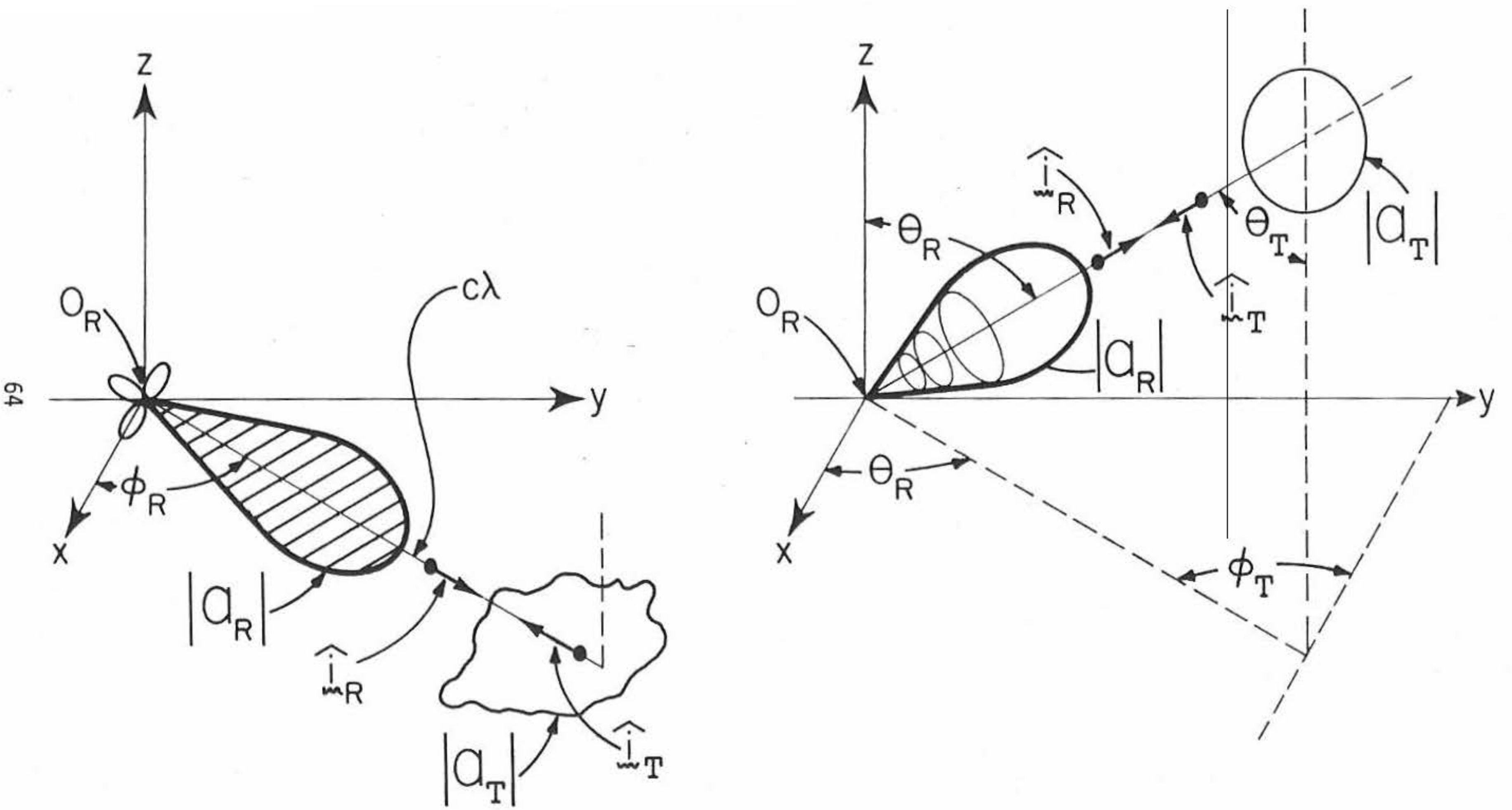

Figure 2.2 (II). Geometries of source and receiver beam patterns: (a), in a plane; (b), in a volume.

$$\begin{cases} A_S = \int_{\Lambda_{S\text{-eff}}} \sigma_S(\lambda)\sigma_S(\phi)c^2\lambda d\lambda d\phi = \int_{[\phi]\text{-eff}} d\phi\sigma_S(\phi)\int_{[\lambda]\text{-eff}} c^2\lambda\sigma_S(\lambda)d\lambda & (2.62a) \\ A_V = \int_{\Lambda_{V\text{-eff}}} \sigma_V(\lambda)\sigma_S(\theta,\phi)c^3\lambda^2\sin\theta\, d\lambda d\theta d\phi & \\ \quad = \int_{(\theta,\phi)\text{-eff}} \sigma_V(\theta,\phi)\sin\theta\, d\theta d\phi \int_{[\lambda]\text{-eff}} c^3\lambda^2\sigma_V(\lambda)d\lambda. & (2.62b) \end{cases}$$

The Λ_{eff} are the effective domains of the possible interfering sources, namely, those capable of registering at our receiver [0_R, cf. Fig. (2.2)II], i.e. observeable in the receiver background noise. The receiver accordingly has a limiting range $c\lambda_{max}$, which depends on $e_{0\gamma}$, cf. (2.34), e.g. $\lambda_{max} = \lambda_{max}(e_{0_\gamma})$. Several cases are distinguished, as shown in Fig.(2.3)II, [as far as dependence on λ is concerned]: From Fig.(2.3)II it is clear that the source domain to be used for $\Lambda_{(SorV)}$ is $\Lambda_{(SorV)\text{-eff}}$: the domain of sources perceivable by the receiver. This is determined by either $\lambda_{eff} = \lambda_{max}$ or λ_Λ [Cases I, II)], whichever is the lesser, or by a pair of λ's, e.g. $\lambda_0 \le \lambda \le \lambda_{max}$, Case III, for such distributions. [Case IV, not shown in Fig.(2.3)II, is a combination of regions ($\pi\lambda_{max}^2$ and Λ) which partially overlap. Here we must consider the overlapping and pertinent nonoverlapping regions separately, which will involve the angles ϕ, or (θ,ϕ) explicitly.] In our present applications, however, we shall assume Case I, e.g. $\lambda_{eff} = \lambda_{max} < \lambda_\Lambda$, which is by far the more prevelant situation in practice: the potential source domain always exceeds that of the receiver's acceptance region.

Finally, we shall, where necessary, postulate the following range dependence of source density:

$$\sigma_{S,V}(\lambda) = 1/\lambda^\mu \quad , \qquad 0 < \mu \quad , \tag{2.63}$$

for both the volume and surface situations, in accordance with our remarks at the beginning of Sec. 2.5.1.

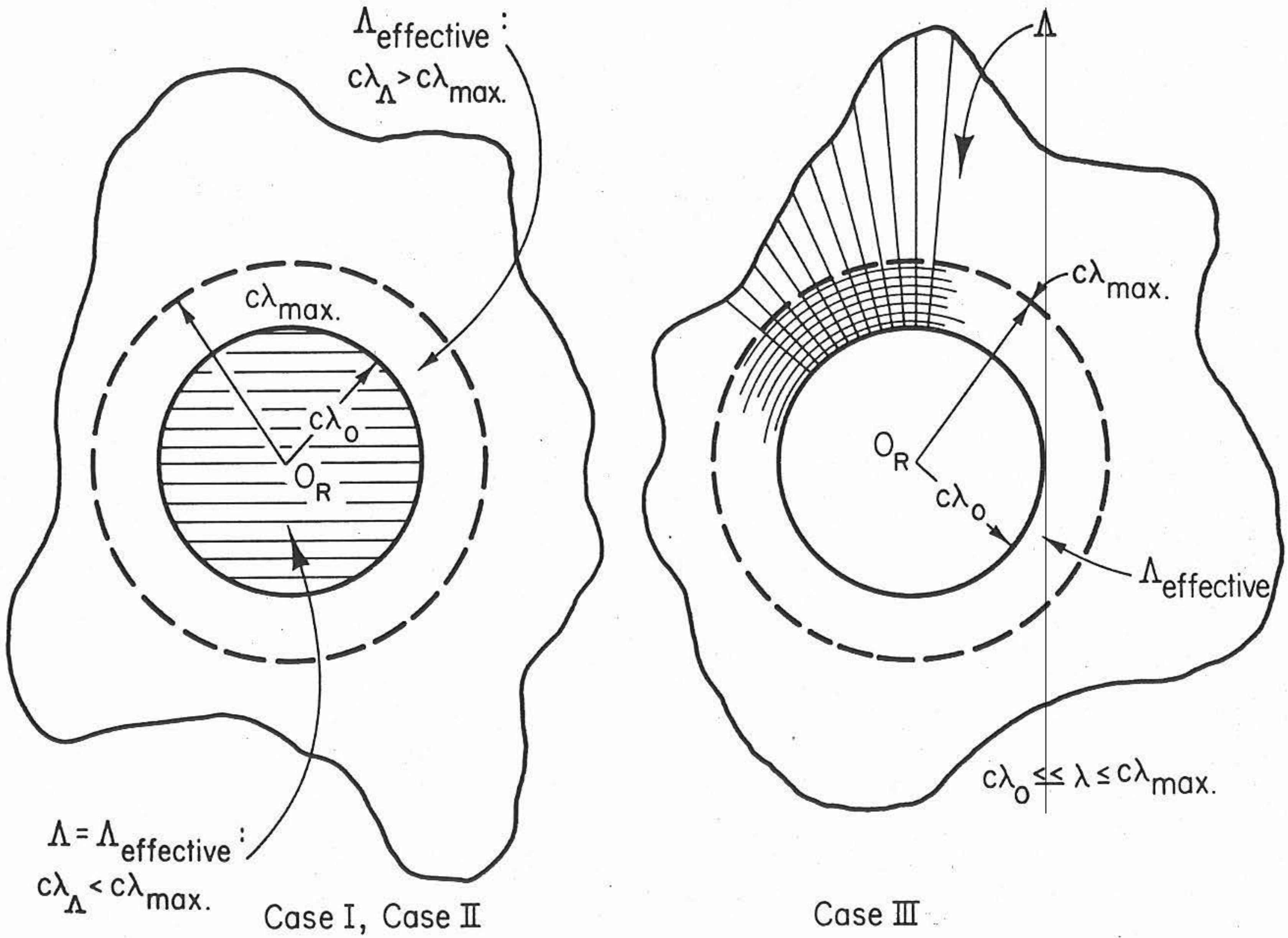

Figure 2.3 (II). Several typical cases (λ only) of source and receiver domains.

2.5.3: Rôle of Input Signal Duration, T_S

We begin by developing in fuller detail the structure of the basic received envelope $\hat{B}_{oA,B}$, cf. (2.41b). Using (2.60), (2.61) in (2.41b) allows us to write

$$\hat{B}_{oA,B} = e_{o\gamma}^{(A,B)} A_o u_{oA,B}(z) | \mathcal{A}_{RT}(\phi;\text{or } \theta,\phi|f_o')| g_{S,V}(\phi;\text{or } \theta,\phi)(4\pi c)^{-\gamma}/\lambda^{\gamma} \tag{2.64a}$$

$$= G_{oA,B}(z,A_o,e_{o\gamma},|\mathcal{A}_{RT}|;\ \phi,\text{or } \theta,\phi)/\lambda^{\gamma}, \tag{2.64b}$$

with

$$G_{oA,B} \equiv e_{o\gamma}^{(A,B)} A_o u_{oA,B} |\mathcal{A}_{RT}| g_{S,V}(4\pi c)^{-\gamma} \tag{2.64c}$$

containing the (possibly) random parameters A_o, $e_{o\gamma}$, $\mathcal{A}_{RT}$, for both surface and volume régimes and Class A and Class B interference. Next, we use (2.62), (2.63) to write (2.59) explicitly as

$$\hat{I}_{\infty}(r)_{A,B} = -A_{\infty}\big|_{A,B} \left\langle \int_0^{z_o,\infty} dz \int_{\Delta(\theta,\phi)} A_{S,V}^{-1}\sigma_{S,V} d(\theta,\phi) \int_{[\lambda]} d\lambda \begin{bmatrix} c^2/\lambda^{\mu-1} \\ c^3/\lambda^{\mu-2} \end{bmatrix} \cdot \int_0^{x=rG_o/\lambda^{\gamma}} J_1(x)dx \right\rangle_{\underset{\sim}{\theta}}, \tag{2.65}$$

with $\underset{\sim}{\theta} = z_o, A_o$, etc., where the upper term applies for surface sources and the lower for those distributed in the volume.

Next, we implement the key step, (iii), in order to interchange the order of integrations over λ and x in (2.65). This permits us to develop $\hat{I}_{\infty}$ explicitly as a function in r, to which we can then apply the approach indicated at the beginning of Section 2.5, to obtain the controlling term(s) at and near $r = 0$ for the characteristic function. Since from (2.59)

$$x = rG_o/\lambda^{\gamma};\ \therefore \lambda = (r,G_o)^{1/\gamma} x^{-1/\gamma}\ ;\ \text{and } \therefore x_o = rG_o/\lambda_{max}^{\gamma} \tag{2.66}$$

is the value of x corresponding to λ_{max}, which establishes the domain of sources perceivable by the receiver [in the present Case I, cf.Fig.(2.3)II]. Now we use the fact that u_{oA} is nonzero only for $(0 \leq z \leq z_o)$, while $u_{oB} \neq 0$, $(0 \leq z < \infty)$. Since $G_{oA,B} \sim u_{oA,B}$, cf. (2.64c), we see at once that for

$$\underline{\text{Class A:}} \quad \begin{cases} 0 \leq x \leq x_o, \text{ since } u_{oA} = 0,\ z > z_o; \\ \\ \lambda_{max} \geq \lambda \geq 0, \text{ cf. Sec. (2.5.2).} \end{cases} \tag{2.67a}$$

$$\underline{\text{Class B:}} \quad \begin{cases} 0 \leq x < \infty, \text{ since } u_{oB} \neq 0,\ z < \infty\ (u_{oB} \to 0,\ z \to \infty); \\ \\ \lambda_{max} \geq \lambda \geq 0, \text{ cf. Sec. (2.5.2).} \end{cases} \tag{2.67b}$$

Fig. (2.4)II shows the allowed domains of x and λ for these two classes of interference. [For Class C interference, we use (2.42), with the c.f.'s of Class A and Class B determined separately, with the help of (2.67) and the results of Section 2.6, 2.7 below. The details are reserved for a subsequent Report.] In Sections (2.6), (2.7) following we obtain the desired c.f.'s (= $\exp(\hat{I}_\infty)$) for these two basic classes of interference, with the help of (2.67) in (2.65) and the observations presented at the beginning of Sec. 2.5.

2.6 The C.F. for Class A Interference:

With Case I conditions (cf. Fig. (2.3)II) on the source distribution vis-à-vis the receiver range (λ_{max}), and (2.67a) applicable here, we see that $x \to x_o$ for the upper limit on the integrand (for x) in (2.65). Accordingly, since $A_{S,V}$ is now precisely equivalent to the indicated integration over (λ,θ,ϕ) therein, we see that (2.65) becomes at once

$$\hat{I}_\infty(r)_A = -A_{\infty,A}\left\langle \int_0^{z_o} dz \int_0^{x_o} J_1(x)dx \right\rangle_{z_o,\underset{\sim}{\theta}} \quad ; \quad x_o = rG_o/\lambda_{max}^{\gamma} = r\hat{B}_{oA} \ . \tag{2.68}$$

Next we use (2.58) to reëxpress (2.68) as sources perceivable by the receiver [in the present Case I, cf.Fig.(2.3)II]. Now we use the fact that u_{oA} is nonzero only for $(0 \leq z \leq z_o)$, while $u_{oB} \neq 0$, $(0 \leq z < \infty)$. Since $G_{oA,B} \sim u_{oA,B}$, cf. (2.64c), we see at once that for

$$\text{\underline{Class A:}} \quad \begin{cases} 0 \leq x \leq x_o, \text{ since } u_{oA} = 0,\ z > z_o; \\ \lambda_{max} \geq \lambda \geq 0, \text{ cf. Sec. (2.5.2).} \end{cases} \tag{2.67a}$$

$$\text{\underline{Class B:}} \quad \begin{cases} 0 \leq x < \infty, \text{ since } u_{oB} \neq 0,\ z < \infty\ (u_{oB} \to 0,\ z \to \infty); \\ \lambda_{max} \geq \lambda \geq 0, \text{ cf. Sec. (2.5.2).} \end{cases} \tag{2.67b}$$

Fig. (2.4)II shows the allowed domains of x and λ for these two classes of interference. [For Class C interference, we use (2.42), with the c.f.'s of Class A and Class B determined separately, with the help of (2.67) and the results of Section 2.6, 2.7 below. The details are reserved for a subsequent Report.] In Sections (2.6), (2.7) following we obtain the desired c.f.'s (= $\exp(\hat{I}_\infty)$) for these two basic classes of interference, with the help of (2.67) in (2.65) and the observations presented at the beginning of Sec. 2.5.

2.6 The C.F. for Class A Interference:

With Case I conditions (cf. Fig. (2.3)II) on the source distribution vis-à-vis the receiver range (λ_{max}), and (2.67a) applicable here, we see that $x \to x_o$ for the upper limit on the integrand (for x) in (2.65). Accordingly, since $A_{S,V}$ is now precisely equivalent to the indicated integration over (λ,θ,ϕ) therein, we see that (2.65) becomes at once

$$\hat{I}_\infty(r)_A = -A_{\infty,A}\left\langle \int_0^{z_o} dz \int_0^{x_o} J_1(x)dx \right\rangle_{z_o,\underset{\sim}{\theta}} \quad ; \quad x_o = rG_o/\lambda_{max}^{\gamma} = r\hat{B}_{oA} \ . \tag{2.68}$$

Next we use (2.58) to reëxpress (2.68) as

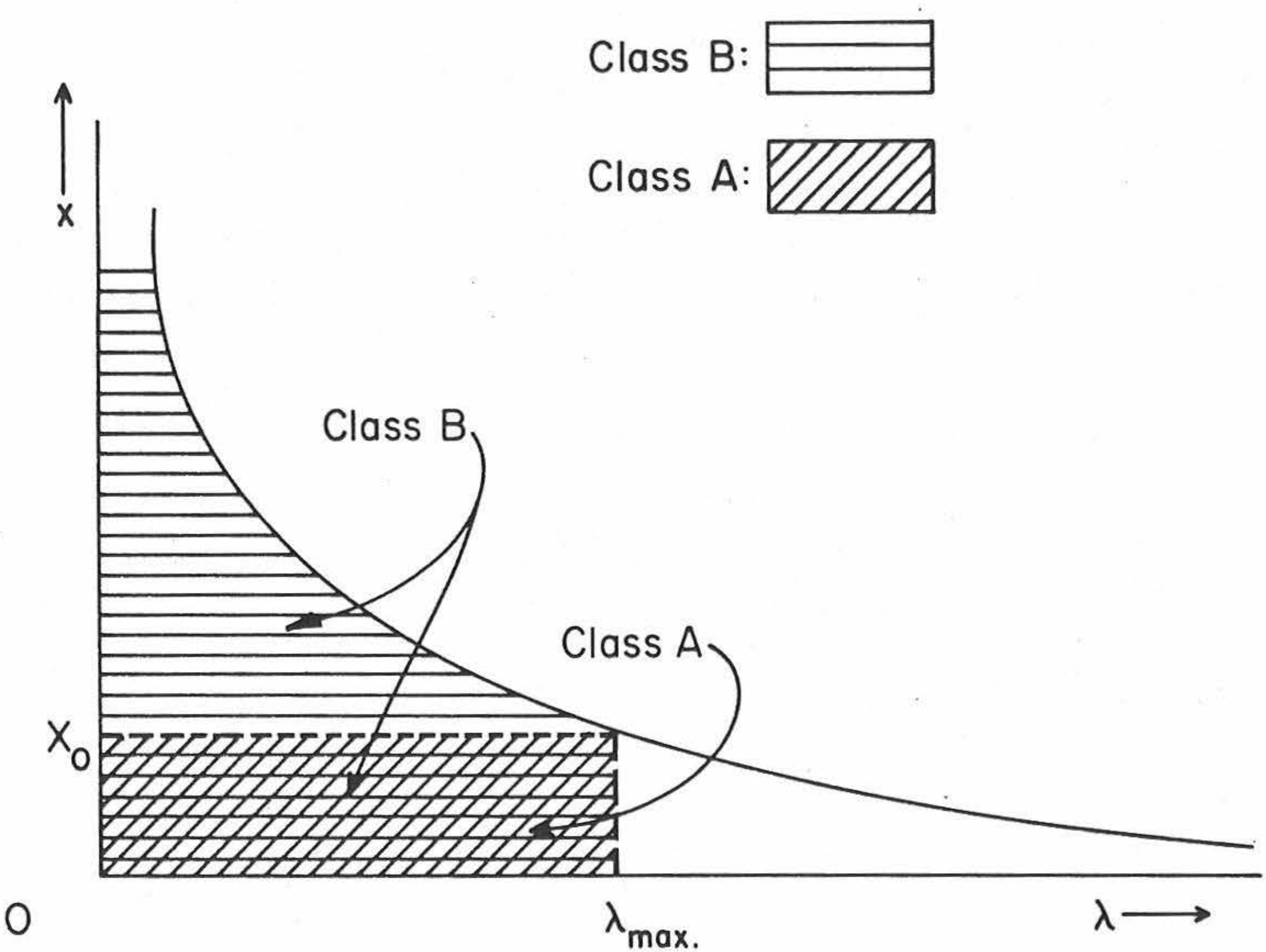

Figure 2.4 (II). Schema of the (x,λ)-domains of intergration for $\hat{I}_\infty$, (2.65), for Class A,B interference.

$$\hat{I}_\infty(r)_A = -A_{\infty,A}\left[1-\left\langle\int_0^{z_0} J_0(x_0)dz\right\rangle_{\underset{\sim}{\theta}}\right], \tag{2.69}$$

since $\bar{z}_0 = \langle T_s\rangle/\bar{T}_s = 1$), and then employ the steepest descent method of Report I [Middleton, 1974, Sec. 3, Eq. (3.10) et seq; as mentioned earlier [cf. Section 2.5)] it is the largest contribution to the exponent in the neighborhood of r=0, which controls the behavior for E >> 0)] to obtain the desired development in r in the neighborhood of r = 0. For this we use the identity

$$\left\langle J_0(x_0)\right\rangle_{\underset{\sim}{\theta}'} = e^{-r^2\langle\hat{B}_{oA}^2\rangle/4}\left\langle J_0(r\hat{B}_{oA})e^{r^2\langle\hat{B}_{oA}^2\rangle/4}\right\rangle_{\underset{\sim}{\theta}'=A_o,\text{etc.}} \; ; \; \underset{\sim}{\theta} = (\underset{\sim}{\theta}', z_0) \tag{2.70}$$

to write

$$\left\langle\int_0^{z_0} J_0(x_0)dz\right\rangle_{\underset{\sim}{\theta}} = e^{-r^2\langle\hat{B}_{oA}^2\rangle/4}\left\langle\int_0^{z_0}\left\langle J_0(r\hat{B}_{oA})\right\rangle_{\underset{\sim}{\theta}'} e^{r^2\langle\hat{B}_{oA}^2\rangle/4} dz\right\rangle_{z_0} \tag{2.71}$$

in (2.69). From (3.11), [Middleton, 1974] we have

$$\left\langle J_0(rA)\right\rangle e^{r^2\langle A^2\rangle/4} = 1 + \sum_{\ell=2}^{\infty}\frac{\langle A^2\rangle^\ell r^{2\ell}}{2^{2\ell}\,\ell!}\left\langle {}_1F_1(-\ell;1;A^2/\langle A^2\rangle)\right\rangle \tag{2.72}$$

where ${}_1F_1(-\ell;1;y^2)$ is given explicitly by (2.56a). The result is that (2.71) becomes

$$\left\langle\int_0^{z_0} J_0(x_0)dz\right\rangle_{\underset{\sim}{\theta}} = e^{-r^2\langle B_{oA}^2\rangle/4}\left\{1 + \sum_{\ell=2}^{\infty}\frac{\langle\hat{B}_{oA}^2\rangle_\theta^\ell r^{2\ell}}{2^{2\ell}\,\ell!}F_\ell\right\}, \tag{2.73}$$

in which

$$F_\ell \equiv \left\langle\int_0^{z_0} {}_1F_1(-\ell;1;\hat{B}_{oA}^2/\langle\hat{B}_{oA}^2\rangle_\theta)dz\right\rangle_{z_0} \; ; \; \langle\hat{B}_{oA}^2\rangle_{\theta=\theta',z_0} = \langle G_0/\lambda_{max}^r\rangle_{\underset{\sim}{\theta}}. \tag{2.73a}$$

Equation (2.73) in (2.69) now gives us the desired expansion of $\hat{I}_{\infty,A}$ about r=0, which governs the p.d. (and P.D.) for large values of the envelope E. We have explicitly

$$\hat{I}_\infty(r)_A = -A_{\infty,A} + A_{\infty,A} e^{-r^2\langle\hat{B}^2_{oA}\rangle/4}\left\{1 + \sum_{\ell=2}^{\infty} \frac{\langle\hat{B}^2_{oA}\rangle^\ell}{2^{2\ell}\,\ell!}\, r^{2\ell} F_\ell\right\} \tag{2.74a}$$

$$\doteq -A_{\infty,A} + A_{\infty,A} e^{-r^2\langle\hat{B}^2_{oA}\rangle/4}\left\{1+O(r^4)\right\}, \tag{2.74b}$$

this last if we neglect the correction terms $[O(r^4$ in $\hat{I}_\infty)]$.

When we consider correction terms [in a later Report] it is convenient to define, as we did in Report I [Middleton, 1974] a set of coefficients, now extended to include the average over z_o:

$$\hat{C}_{2\ell} \equiv \ell!(-1)^\ell\left\langle\int_0^{z_o} {}_1F_1(-\ell;1;\hat{B}^2_{oA}/\langle\hat{B}^2_{oA}\rangle)dz\right\rangle_{z_o} = (-1)^\ell \ell!\, F_\ell, \tag{2.75}$$

cf. (2.73a), which contain the 2ℓ, $2\ell-2, \ldots,$ moments of the envelope $\hat{B}_{oA}$ at the output of the ARI stages of the receiver. Specifically, we have here

$$\hat{C}_4 = [\langle\hat{B}^4_{oA}\rangle - 2\langle\hat{B}^2_{oA}\rangle^2]/\langle\hat{B}^2_{oA}\rangle^2 \quad ; \ (\hat{C}_2 = 0) \tag{2.75a}$$

$$\hat{C}_6 = [\langle\hat{B}^6_{oA}\rangle - 9\langle\hat{B}^4_{oA}\rangle\langle\hat{B}^2_{oA}\rangle + 12\langle\hat{B}^2_{oA}\rangle^3]/\langle\hat{B}^2_{oA}\rangle^3 \ ; \tag{2.75b}$$

$$\hat{C}_8 = [\langle\hat{B}^8_{oA}\rangle - 16\langle\hat{B}^6_{oA}\rangle\langle\hat{B}^2_{oA}\rangle + 72\langle\hat{B}^4_{oA}\rangle\langle\hat{B}^2_{oA}\rangle^2 - 72\langle\hat{B}^2_{oA}\rangle^4]/\langle\hat{B}^2_{oA}\rangle^4; \text{ etc.,} \tag{2.75c}$$

with

$$\langle\hat{B}^{2n}_{oA}\rangle \equiv \left\langle\int_0^{z_o}\langle\hat{B}^{2n}_{oA}\rangle_{\theta'} dz\right\rangle_{z_o} = \left\langle\int_0^{z_o}\langle G^{2n}_{oA}\rangle_{\underset{\sim}{\theta}'}\lambda^{2\gamma n}_{max} dz\right\rangle_{z_o} . \tag{2.75d}$$

Thus, (2.74) becomes here, equivalently,

$$\hat{I}_\infty(r)_A = -A_{\infty,A} + A_{\infty,A} e^{-r^2\langle\hat{B}^2_{oA}\rangle/4}\left\{1 + \sum_{\ell=2}^{\infty}\frac{\langle\hat{B}^2_{oA}\rangle^\ell(-1)^\ell}{2^{2\ell}(\ell!)^2}\,\hat{C}_{2\ell} r^{2\ell}\right\}. \tag{2.76}$$

Combining (2.74), or (2.76), with the practical situation involving an additive background gaussian component, cf. Sec. (2.3.1), we get finally for the desired form of the c.f. for Class A interference

$$\hat{I}_\infty(r)_{A+G} = -\sigma_G^2 r^2/2 + \hat{I}_\infty(r)_A ,$$

so that

$$\boxed{\hat{F}_1(ir)_{A+G} = \exp\left\{-\sigma_G^2 r^2/2 - A_{\infty,A} + A_{\infty,A} e^{-r^2\langle\hat{B}_{oA}^2\rangle/4}\left[1+O(r^4)\right]\right\} ,} \tag{2.77}$$

where $[1+O(r^4)]$ in (2.77) is given explicitly by the expression { } in (2.76). The final step in the reduction of these c.f.'s to the desired technically manageable form, particularly for the smaller values $O(\leq 10^0)$ of the impulsive index $A_{\infty,A}$, (2.39), which are typical of this class of interference, is now the following direct expansion of the terms containing $A_{\infty,A}$ in (2.76) or (2.77):

$$\boxed{\begin{aligned}\hat{F}_1(ir)_{A+G} = {} & e^{-A_{\infty,A}} \sum_{m=0}^{\infty} \frac{A_{\infty,A}^m}{m!} e^{-[m\langle\hat{B}_{oA}^2\rangle/2+\sigma_G^2]r^2/2} \\ & \cdot \left\{1 + \frac{A_{\infty,A}\langle\hat{B}_{oA}^2\rangle r^4 \hat{C}_4}{4^3} e^{-r^2\langle\hat{B}_{oA}^2\rangle/4} \right. \\ & \left. + \frac{A_{\infty,A}\langle\hat{B}_{oA}^2\rangle^3 r^6 \hat{C}_6}{4^3(3!)^2} e^{-r^2\langle\hat{B}_{oA}^2\rangle/4} + \dots\right\}\end{aligned}} \tag{2.78}$$

[Equation (2.78) is formally the same as our earlier result (3.16) [Middleton, 1974], which is not surprising, since the c.f.'s have the same form, cf. (2.50). The parameters differ somewhat in detail, of course.]

However, in approximating the exact (2.74a) c.f. by the approximate forms (2.74b), (2.77), (2.78), we obtain ultimately (cf. Section 3 following) approximate P.D.'s, and pdf's (Section 4, ff), which may not be properly normalized, in the sense of yielding a mean square value, $\langle E^2 \rangle_A$, of the envelope, different from the exact relation $\langle E^2 \rangle_A = 2\Omega_{2A}(1+\Gamma_A')$, cf. Eq. (5.12b), which is derived from the exact c.f. (5.10), or (5.10a).

Accordingly, we choose the c.f., here and subsequently the leading term of (2.78), and from it determine the associated P.D. and pdf. Then, from the pdf we determine

$$\langle \mathcal{E}^2 \rangle_A = \int_0^\infty w_1(\mathcal{E})_A \mathcal{E}^2 d\mathcal{E} \ , \quad \mathcal{E} \equiv \frac{E}{\sqrt{2\Omega_{2A}(1+\Gamma_A')}} \ . \tag{2.78a}$$

If $w_1(\mathcal{E})_A$ is properly normalized, $\langle \mathcal{E}^2 \rangle_A$ should be unity. If not, and $\langle \mathcal{E}^2 \rangle_A$ is bounded, such that $\langle \mathcal{E}^2 \rangle_A \equiv N_A^2$ $(<\infty, \neq 1)$, then the properly normalized pdf ($\therefore$ PD, also) is given by

$$w_1(\mathcal{E})_{A=\text{norm}} = w_1(\mathcal{E})_A N_A^{-2} \tag{2.78b}$$

so that

$$\langle \mathcal{E}^2 \rangle_{(A)} = \int_0^\infty w_1(\mathcal{E})_A N_A^{-2} \mathcal{E}^2 d\mathcal{E} = \int_0^\infty w_1(\mathcal{E})_{A\text{-norm}} \mathcal{E}^2 d\mathcal{E} = 1 \ , \tag{2.78c}$$

as required. From (5.3) following we have (cf. Sec. 5.1):

$$\langle \mathcal{E}^2 \rangle_A = e^{-A_{\infty,A}} \sum_{m=0}^{\infty} \left(\frac{m/A_{\infty,A}+\Gamma_A'}{1+\Gamma_A'}\right) \frac{A_{\infty,A}^m}{m!} = 1 \tag{2.78d}$$

exactly, showing that $w_1(\mathcal{E})_A$ here is indeed suitably normalized (and $\therefore N_A^2 = 1$), as is the PD, $P_1(\mathcal{E} > \mathcal{E}_0)_A$, consequently. Accordingly, the Class A cases discussed subsequently need no scale or other adjustments (unlike the Class B model, where the approximating forms do require scale and level adjustments, cf. remarks following Eq. (2.94), and the discussion in

Section 3.2-A.

An interesting feature of our results here, for Class A interference, is that the c.f., and hence the p.d., and P.D., are not explicitly dependent on the interference source distribution [for the usual Case I, II, and not so common Case III source-receiver conditions, cf. Fig. (2.3)II]. Furthermore, these statistics are insensitive to the propagation law ($\sim \lambda_{max}^{-\gamma}$) involved, which merely affects scale, through the average $\langle \hat{B}_{0A}^2 \rangle = \langle G_0/\lambda^\gamma \rangle$. The source distribution does appear, but in averaged form and only in the impulsive index $A_{\infty,A}$, [cf. (2.35), (2.38), (2.39), in conjunction with (2.27)]. Physically, this is understandable, since it is only the average number of emissions per second times the mean duration of these finite emissions (in the receiver, cf. Fig. (2.1)II) to which the receiver can respond. It has no way to distinguish where and in what concentration, or by what propagation law, the sources may be acting [for a given position of the receiver beam $\mathcal{Q}_R$, or for any position if $\mathcal{Q}_R$ is omnidirectional]. The only thing that counts here in determining the (first-order) statistics of the received input is total input level and process "density", $A_{\infty,A}$. As we shall see in Section (2.7), and in later Sections, this insensitivity to source distribution and propagation law is definitely not characteristic of Class B interference, and, consequently, Class C noise, to the extent that its Class B component is significant.

2.7 The C.F. for Class B Interference:

Here we use (2.67b) for the exponent (2.65). The result is a term like (2.68), plus an additional term for $x_0 < x < \infty$, with $\lambda_{max} \geq \lambda \geq 0$, viz:

$$\hat{I}_\infty(r)_B = -A_{\infty,B}\left\{\left\langle \int_0^\infty dz \int_0^{x_0=rG_0/\lambda_{max}^\gamma} J_1(x)\,dx \right\rangle_{\underset{\sim}{\theta}'} + \left\langle \int_0^\infty dz \int_{\Delta(\theta,\phi)} A_{S,V}^{-1}\sigma_{S,V}(\theta,\phi)\,d(\theta,\phi) \int_{x_0=rG_0/\lambda_{max}^\gamma}^\infty J_1(x)\,dx \cdot \int_{\lambda=(rG_0)^{1/\gamma}/x^{1/\gamma}}^{0} \begin{bmatrix} c^2/\lambda^{\mu-1} \\ c^3/\lambda^{\mu-2} \end{bmatrix} d\lambda \right\rangle_{\underset{\sim}{\theta}'} \right\}, \tag{2.79}$$

Section 3.2-A.

An interesting feature of our results here, for Class A interference, is that the c.f., and hence the p.d., and P.D., are not explicitly dependent on the interference source distribution [for the usual Case I, II, and not so common Case III source-receiver conditions, cf. Fig. (2.3)II]. Furthermore, these statistics are insensitive to the propagation law ($\sim \lambda_{max}^{-\gamma}$) involved, which merely affects scale, through the average $\left\langle \hat{B}_{oA}^2 \right\rangle = \left\langle G_o/\lambda^\gamma \right\rangle$. The source distribution does appear, but in averaged form and only in the impulsive index $A_{\infty,A}$, [cf. (2.35), (2.38), (2.39), in conjunction with (2.27)]. Physically, this is understandable, since it is only the average number of emissions per second times the mean duration of these finite emissions (in the receiver, cf. Fig. (2.1)II) to which the receiver can respond. It has no way to distinguish where and in what concentration, or by what propagation law, the sources may be acting [for a given position of the receiver beam $\mathcal{A}_R$, or for any position if $\mathcal{A}_R$ is omnidirectional]. The only thing that counts here in determining the (first-order) statistics of the received input is total input level and process "density", $A_{\infty,A}$. As we shall see in Section (2.7), and in later Sections, this insensitivity to source distribution and propagation law is definitely not characteristic of Class B interference, and, consequently, Class C noise, to the extent that its Class B component is significant.

2.7 The C.F. for Class B Interference:

Here we use (2.67b) for the exponent (2.65). The result is a term like (2.68), plus an additional term for $x_o < x < \infty$, with $\lambda_{max} \geq \lambda \geq 0$, viz:

$$\hat{I}_\infty(r)_B = -A_{\infty,B}\Bigg\{\left\langle \int_0^\infty dz \int_0^{x_o = rG_o/\lambda_{max}^\gamma} J_1(x)dx \right\rangle_{\underset{\sim}{\theta}'}$$
$$+\Bigg\langle \int_0^\infty dz \int_{\Delta(\theta,\phi)} A_{S,V}^{-1} \sigma_{S,V}(\theta,\phi) d(\theta,\phi) \int_{x_o = rG_o/\lambda_{max}^\gamma}^\infty J_1(x)dx$$
$$\cdot \int_{\lambda=(rG_o)^{1/\gamma}/x^{1/\gamma}}^0 \begin{bmatrix} c^2/\lambda^{\mu-1} \\ c^3/\lambda^{\mu-2} \end{bmatrix} d\lambda \Bigg\rangle_{\underset{\sim}{\theta}'} \Bigg\}, \tag{2.79}$$

where we have used (2.66) and reversed the order of integration, according to the régime of (iii), Sec. 2.5, and Fig.2.4II) above. Note, in particular, the order of the limits on the λ-integration, which correspond to the variation in x, from $x=x_0$ to $x\to\infty$, cf. Fig.(2.4)II again. [The average over z_0 is unity, as none of the arguments contain z_0; also, $\underset{\sim}{\theta}' = A_0$, $e_{0\gamma}$, etc., as before.] The integrals I_ϕ, $I_{\theta,\phi}$ over ϕ (or θ,ϕ), with $A^{-1}_{S,V}$, become explicitly from (2.62a,b)*

$$\left.\begin{aligned} I_\phi &= \Delta_S/A_S = \left(\frac{2-\mu}{c^2}\right)\lambda_{max}^{\mu-2}, \quad 0 \leq \mu < 2 \\ I_{\theta,\phi} &= \Delta_V/A_V = \left(\frac{3-\mu}{c^3}\right)\lambda_{max}^{\mu-3}, \quad 0 \leq \mu < 3 \end{aligned}\right\}, \tag{2.80}$$

where specifically here

$$A_S = \Delta_S \frac{c^2}{2-\mu}\lambda_{max}^{2-\mu} \quad (0 < \mu < 2); \quad A_V = \Delta_V \frac{c^3}{3-\mu}\lambda_{max}^{3-\mu} \quad (0 < \mu < 3), \tag{2.80a}$$

so that Δ_S, Δ_V are respectively the integrals $\int \sigma_S d\phi$, $\int \sigma_V d\theta d\phi$ in (2.79).

With the above we readily find for (2.79) the (exact) relation

$$\boxed{\hat{I}_\infty(r)_B = -A_{\infty,B}\left\{\left\langle\int_0^\infty dz\int_0^{x_0} J_1(x)dx\right\rangle_{\underset{\sim}{\theta}'} - x_0^\alpha\left\langle\int_0^\infty dz\int_{x_0}^\infty J_1(x)dx/x^\alpha\right\rangle_{\underset{\sim}{\theta}'}\right\},} \tag{2.81}$$

with

$$x_0 = rG_0/\lambda_{max}^\gamma = r\hat{B}_{0,B} \tag{2.81a}$$

as before, and the new parameter

---

* For this Report we shall limit the allowed values of μ as shown in (2.80); extension to other values (μ>2,3) will be considered in a subsequent study, as is the analysis for $\alpha\geq 2$.

$$\alpha \equiv \left.\frac{2-\mu}{\gamma}\right|_{\text{surface}} \quad \text{or} \quad \left.\frac{3-\mu}{\gamma}\right|_{\text{vol.}} \quad . \tag{2.82}$$

This parameter α we call the spatial density-propagation parameter, since it depends on the interacting spatial effects of source density and source propagation law.

The lower limit on α is established by the present condition, i.e., upon the upper limit on μ (=2,or 3), $\gamma > 0$. Analytically, for the integral over x in (2.81) to be convergent, we require that $\alpha > -1/2$. There is, however, no (finite) upper limit on α, so that we can write

$$\boxed{-1/2 < \alpha < \infty \quad .} \tag{2.82a}$$

For the purposes of the present Report we shall, however, further restrict α to the range $(0<\alpha<2)$, which covers many of the practical cases encountered in applications, at least down to quite small values of the exceedance probabilities $P_1(\mathcal{E} < \mathcal{E}_0)$. In a later Report we shall develop the analysis in detail for $(\alpha \geq 2)$.

The first integral in (2.81) is readily evaluated by expanding the Bessel function, followed by termwise integration. We get

$$I_1 = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} \left\langle x_0^{2\ell+2} \right\rangle_{z,\underset{\sim}{\theta}'}}{\ell!(\ell+1)!2^{2\ell+1}(2\ell+2)} \quad . \tag{2.83}$$

For the second integration, over x in the second integral of (2.81), we use a Barnes integral representative of $J_1(x)$ [cf. Middleton, 1960, Eq. (13.10)]:

$$J_1(x) = \int_{-\infty i-c}^{\infty i-c} \frac{\Gamma(-s)}{\Gamma(s+2)} \left(\frac{x}{2}\right)^{2s+1} \frac{ds}{2\pi i} \quad , \quad c > 0 \ , \tag{2.84}$$

so that

$$I_\alpha \equiv \left\langle x_0^\alpha \int_{x_0}^\infty \frac{J_1(x)}{x^\alpha}\,dx \right\rangle_{z,\underset{\sim}{\theta}'} = \left\langle x_0^\alpha \int_{-\infty i-c}^{\infty i-c} \frac{\Gamma(-s)ds/2\pi i}{\Gamma(s+2)2^{2s+1}} \int_{x_0}^\infty x^{2s+1-\alpha}dx \right\rangle_{z,\underset{\sim}{\theta}'} . \tag{2.85a}$$

This becomes, on choosing $c = -5/4$ (as a result of the condition $\alpha > -1/2$) and carrying out the x-integration

$$\left.\frac{x^{2s+2-\alpha}}{2s+2-\alpha}\right|_{x_0}^{\infty} = \frac{-x_0^{2s+2-\alpha}}{2s+2-\alpha} \; ; \begin{cases} \mathrm{Re}(2s+2-\alpha) < 0, \text{ or} \\ s < \frac{\alpha-2}{2} . \end{cases} \tag{2.85b}$$

The resulting integral I_α is now explicitly

$$I_\alpha = \left\langle x_0^\alpha \int_{-\infty i-c}^{\infty i-c} \frac{-\Gamma(-s)x_0^{2s+2-\alpha}ds/2\pi i}{\Gamma(s+2)2^{2s+1}(2s+2-\alpha)} \right\rangle_{z,\underset{\sim}{\theta}'} , \tag{2.85c}$$

which has a simple pole at $s = (\alpha-2)/2$ $(>-5/4)$, and at $s=0,1,2,3,\ldots$ $(0 < \alpha < 2)$. The residue at $s = (\alpha-2)/2$ is $-\Gamma(1-\alpha/2)/2^{\alpha-1}\Gamma(1+\alpha/2)$, while those of Γ-function are $(-1)^\ell/\ell!$ at $s=\ell$ $(=0,1,2,..)$. The result is

$$I_\alpha = \frac{-\Gamma(1-\alpha/2)\langle x_0^\alpha\rangle}{2^{\alpha-1}\Gamma(1+\alpha/2)} - \sum_{\ell=0}^{\infty} \frac{(-1)\;\langle x_0^{2\ell+2}\rangle}{\ell!(\ell+1)\;2^{2\ell+1}(2\ell+2-\alpha)} . \tag{2.85d}$$

The exponent $\hat{I}_\infty$ of the c.f. for Class B interference thus becomes, on combining I_1, (2.83), and I_α, (2.85d), in (2.81):

$$\hat{I}_{\infty,B}(r) = -A_{\infty,B}\left\{\frac{\Gamma(1-\alpha/2)\langle x_0^\alpha\rangle}{2^{\alpha-1}\Gamma(1+\alpha/2)} + \sum_{\ell=0}^{\infty}\frac{(-1)^\ell\langle x_0^{2\ell+2}\rangle}{\ell!(\ell+1)!2^{2\ell+1}}\left[\frac{4\ell+4-\alpha}{(2\ell+2)(2\ell+2-\alpha)}\right]\right\}, \quad (0 < \alpha < 2) , \tag{2.86}$$

with $\langle\ \rangle = \langle\ \rangle_{z,\underset{\sim}{\theta}'} \equiv \langle \int_0^\infty(\ )dz\rangle_{\theta'}$, etc.

With the additive, accompanying gaussian background, cf. Sec. (2.3.1), we have at last the desired c.f.:

$$\boxed{\hat{F}_1(ir)_{B+G} = \exp\left\{-b_{1\alpha}A_{\infty,B}r^{\alpha} - (\sigma_G^2 + b_{2\alpha}A_{\infty,B})r^2/2 - \sum_{\ell=1}^{\infty}(-1)^{\ell} b_{(2\ell+2)\alpha}A_{\infty,B}r^{2\ell+2}\right\}, (0 < \alpha < 2),} \quad (2.87)$$

which, like (2.81), is also exact, so far. Here we have explicitly

$$\begin{cases} b_{1\alpha} \equiv \dfrac{\Gamma(1-\alpha/2)}{2^{\alpha-1}\Gamma(1+\alpha/2)} \left\langle G_{o,B}^{\alpha}\right\rangle \lambda_{max}^{-\alpha\gamma} = \dfrac{\Gamma(1-\alpha/2)}{2^{\alpha/2-1}\Gamma(1+\alpha/2)} \left\langle\left(\dfrac{\hat{B}_{o,B}}{\sqrt{2}}\right)^{\alpha}\right\rangle (>0) & (2.87a) \\[2ex] b_{2\alpha} \equiv \left(\dfrac{4-\alpha}{2-\alpha}\right) \left\langle G_{o,B}^2\right\rangle \lambda_{max}^{-2} = \left(\dfrac{4-\alpha}{2-\alpha}\right) \dfrac{\left\langle \hat{B}_{o,B}^2\right\rangle}{2} \quad (>0), & (2.87b) \\[2ex] b_{(2\ell+2)\alpha} \equiv \dfrac{(4\ell+4-\alpha)}{\ell!(\ell+1)!(2\ell+2-\alpha)(2\ell+2)} \dfrac{\left\langle \hat{B}_{o,B}^{2\ell+2}\right\rangle}{2^{2\ell+1}} \quad (>0); \ (\hat{B}_{o,B} = G_{o,B}/\lambda_{max}^{\gamma}), & (2.87c) \end{cases}$$

since $-0 < \alpha < 2$, and from (2.65) we write

$$\left\langle G_{o,B}^{\alpha}\right\rangle = \left\langle e_{o\gamma}^{(B)\alpha}\right\rangle \left\langle A_o^{\alpha}\right\rangle \left\langle |\mathcal{Q}_{RT}|^{\alpha}\right\rangle_{\mathcal{Q}} \left\langle g_{S,V}^{\alpha}\right\rangle (4\pi c)^{-\alpha\gamma} \int_0^{\infty} u_o(z)_B^{\alpha} dz \quad (>0), \quad (2.87d)$$

and formally $\left\langle G_{o,B}^{2\ell+2}\right\rangle$ is given by (2.87d) on replacing α by $2\ell+2$, etc.

A. The Approximating C.F.'s for $(0 < \alpha < 2)$:

Unlike Class A interference [Sec. 2.6], where the c.f. is solely a function of r^2 [cf. (2.76)-(2.78)], and where a single "steepest-descent" approximation [cf.(2.72)-(2.76)] provides a good fit for both large, small, and intermediate values of r^2 (and hence for E_0,E), Class B noise requires a pair of approximating c.f.'s, one of which will at least insure suitably bounded behaviour of the exceedance probability P_1 $(E > E_0)$ as $E_0 \to \infty$, including the existence of all finite moments of the envelope $\langle E^{\beta}\rangle$ $(0<\beta<\infty)$, and the other of which will provide a satisfactory account of P_1 for small and intermediate values of E ($>E_0$). It is the presence of the term $O(r^\alpha)$ in the (exponent of the) c.f. (2.87) for Class B interference, in addition to the typical development in powers of r^2 (analogous to that for the Class A noise), which forces this double approximation for our canonical c.f.'s, and P.D.'s, $P_1(E>E_0)$, pdf's, $w_1(E)$, here.

At this point we define the gaussian variance

$$\Delta\sigma_G^2 \equiv \sigma_G^2 + b_{2\alpha}A_{\infty,B} = b_{2\alpha}A_{\infty,B}(1+\sigma_G^2/b_{2\alpha}A_{\infty,B}) \tag{2.88a}$$

$$\equiv \Omega_{2B}^{(G)}(1+\Gamma_B^{(G)}), \tag{2.88b}$$

with

$$\Omega_{2B}^{(G)} \equiv b_{2\alpha}A_{\infty,B} \quad ; \Gamma_B^{(G)} \equiv \sigma_G^2/\Omega_{2B}^{(G)} = \frac{\text{indep. gauss intensity}}{\text{"impulsive" gauss Intensity}} \tag{2.88c}$$

$$= \left(\frac{4-\alpha}{2-\alpha}\right)\Omega_{2B}, \text{ cf. Eq. (3.2a)ff.}$$

where $\Omega_{2B}^{(G)}$ is the "impulsive" contribution to the gaussian component arising from the Class B noise alone, and where $\sigma_G^2(=\sigma_E^2+\sigma_R^2)$ are the (independent) inherently gaussian contributions from potential external (gaussian) sources and from the receiver noise (essentially all arising in the initial linear input stages), cf.(2.47). (Note that $\Omega_{2B}^{(G)},\Gamma_B^{(G)}$ are also functions of α here.)

For the c.f. which is appropriate to the intermediate range of envelope values, including the very small ($E,E_0\to 0$), the controlling term in the exponent of the (exact) c.f. (2.87) is <u>the smallest power of r with negative coefficient</u>, e.g., $-b_{1\alpha}A_{\infty,B}r^\alpha$ here, so that this approximate form remains a proper c.f., e.g. $\lim_{r\to 0}\hat{F}_1 = 1$, $\lim_{r\to\infty}\hat{F}_1\to 0$. The form of the associated pdf. and P.D. for small and moderate values of E, E_0 is governed principally by the behaviour of the c.f. as r becomes large. Thus, as a first approximation which ignores any gaussian contributions, we have from (2.87)

$$\boxed{\hat{F}_1(ir)'_B \doteq e^{-b_{1\alpha}A_{\infty,B}r^\alpha} \quad , \qquad 0 < \alpha < 2.} \tag{2.89}$$

However, practically there is always at least an observeable gaussian system noise component, and as noted above, cf. (2.87), (2.88a), an additional gaussian term ($\sim b_{2\alpha}A_{\infty,B}$) contributed by the "impulsive" Class B noise, so that the more realistic intermediate c.f. here is now

$$\boxed{\hat{F}_1(Ir)_{(B+G)-I} \doteq e^{-b_{1\alpha}A_{\infty,B}r^{\alpha}-\Delta\sigma_G^2 r^2/2}} \quad , \ (0 < \alpha < 2), \qquad (2.90)$$

where the subscript (-I) indicates the c.f. for the range ($0 \leq E, E_0 \leq E_B$) of envelope values. (The precise definition of E_B will be given presently, cf. Sec. 3.2).

For values of $E, E_0 > E_B$ we require a c.f. approximating the exact relation (2.87) where the <u>largest</u> (r-dependent) <u>contribution to the exponent about r = 0 and in the finite</u> (nonzero) <u>neighborhood of r=0</u> is the controlling term. For this we seek again a "steepest descent" form for the exponent of (2.87), exclusive of the term in r^{α}, which as we shall see below is always here smaller than the former (for $0 \leq r \leq \varepsilon$) and thus does not control the character of the c.f. at small r (and hence for large E, E_0). Accordingly, as in the Class A cases, cf. Section 2.6 above, we wish to represent the class B terms (exclusive of r^{α}) in (2.87) by a series of the form

$$A_{\infty,B}-b_{2\alpha}A_{\infty,B}\frac{r^2}{2}+\sum_{\ell=1}^{\infty}(-1)^{\ell+1}A_{\infty,B}b_{(2\ell+2)\alpha}r^{2\ell+2} = Ae^{-ar^2}\left[1+\sum_{k=1}^{\infty}B_k r^{2k}\right], \qquad (2.91)$$

where the "steepest-descent" nature of the approximation is exhibited not only by the exponential factor but by requiring the vanishing of the B_1-term in the right hand series, where the nearest "correction" term (k=2) is $O(r^4)$ and quite ignorable vis-à-vis unity. This condition and a term by term comparison of (2.91) determine all the parameters A, a, B_k ($k \geq 2$), which are readily found to be

$$A = A_{\infty,B} \; ; \; a = b_{2\alpha}/2 \quad ; \quad \left.\begin{array}{l} (B_1 = 0) \\ B_2 = b_{4\alpha} - b_{2\alpha}^2/8 \\ B_3 = b_{6\alpha} + \dfrac{b_{4\alpha} b_{2\alpha}}{2} + \dfrac{b_{2\alpha}^3}{48} \;, \text{ etc.} \end{array}\right\} . \tag{2.92}$$

Clearly, $A_{\infty,B}e^{-ar^2}$ dominates $-A_{\infty,B}b_{1\alpha}r^{\alpha}$, $A_{\infty,B}e^{-ar^2}r^4$, etc., at and in the neighborhood of $r = 0$, and this is the determining element for this approximation to the exact c.f. (2.87).

Accordingly, we have finally for the c.f. appropriate at least to the large values of E, E_0, i.e. for the "rare events",

$$\boxed{\hat{F}_1(ir)_{(B+G)-II} \doteq \left\{ e^{-A_{\infty,B}} \exp\left[A_{\infty,B} e^{-b_{2\alpha} r^2/2} - \sigma_G^2 r^2/2 \right] \right\} [1+O(r^{\alpha}, r^4)], \quad (0 < \alpha < 2).} \tag{2.93}$$

Comparison with (2.77) shows at once that this approximate c.f. for Class B interference has the same (approximate) form as that for Class A noise, and thus will yield the same type of pdf's and P.D.'s, etc., cf. Sec. (3.1). As we shall see later, in Section 4, this has the important consequence of insuring that all (finite) moments of the envelope, $\langle E^{\beta} \rangle$, exist, as required by the physics of the situation in all cases.

We note in passing that a more elaborate approximation to $\hat{F}_1$ here may be obtained by a combination of (2.89) and (2.93), viz:

$$\hat{F}_1(ir)_{(B+G)-III} = e^{-b_{1\alpha}A_{\infty,B}r^{\alpha} - A_{\infty,B} - \sigma_G^2 r^2/2} \exp\left(A_{\infty,B} e^{-b_{2\alpha} r^2/2} \right) , \tag{2.94}$$

which may be used for intermediate ranges of r for improved fits to the corresponding intermediate ranges for the envelope. However, since the resulting pdf's, and P.D.'s, are considerably more complex and since the simpler forms of c.f. above, e.g. (2.90), (2.93), appear ultimately to provide excellent agreement with observation, we shall not pursue the consequences of using (2.94) further in the present Report.

We remark that both approximating c.f.'s (2.90), (2.93) for the true Class B c.f. are such as to give pdf's which are not properly normalized; each pdf, $w_1(\mathcal{E})_{B-I}, w(\mathcal{E})_{B-II}$, (4.3),(4.4) does <u>not</u> yield $\langle\mathcal{E}^2\rangle_B = 1$. The former gives an infinite value, while the latter, although Class A-type, cf. (2.78), yields $\langle\mathcal{E}^2\rangle_{B-II} = 4G_B^2$ ($\neq 1$), where G_B is given by (3.12b). Thus, $w_1(\mathcal{E})_{B-II,norm} = (4G_B^2)^{-1} w_1(\mathcal{E})_{B-II}$, while the normalization of $w_1(\mathcal{E})_{B-I}$ requires, instead, a change of scale for the argument $\mathcal{E}$ (and $\therefore$ $\mathcal{E}_0$ in the associated PD). How this is done is described in Section 3.2-A.

Finally, it is important to observe that unlike the Class A interference discussed in Section 2.6 above, the (first-order) statistics of Class B noise are obviously sensitive to the combined effects of source distribution (μ) and propagation law (γ), through the density-propagation parameter α, cf. (2.82). Physically, this may generally be explained by the fact that now the receiver itself largely determines the waveform of its response to the (relatively) short input excitations, unlike Class A noise, where apart from amplification (for fixed aperture bearing) the receiver negligibly influences the structure of the received wave trains. The composite sum of the "tails" of the transients in the receiver, generated by the Class B input, depend on the (relative) times of arrival of individual wave trains ($\sim$source distribution) and on the level of the various wavetrains ($\sim$source distribution and propagation law). The (relatively) longer time-pedestal provided by the transient decay of individual impulses provides a wider range of possible total amplitudes of overlapping transients and hence a more gradual transition to given thresholds (E_0) of the exceedance probabilities $P_1(E > E_0)$, than that occuring with Class A interference, as can be seen subsequently in Figs.(3.5, 3.6)II vs. Figs.(3.1)II,(3.2)II. [These effects accordingly influence the instantaneous waveform in the receiver's ARI stage, and hence the statistics of that waveform.] In any case, the sensitivity to α is thus a receiver bandwidth phenomenon, which is illustrated by the experimental and theoretical results shown in Part I of this Report.

3. PROBABILITY DISTRIBUTIONS: $P_1(\mathcal{E} > \mathcal{E}_0)_{A,B}$, $(0 < \alpha < 2)$

We are now ready to obtain the first-order exceedance probabilities $P_1(E > E_0)$, cf. (2.22b), when an independent gaussian component is present, so that Eqs. (2.78), and (2.90), (2.93) apply respectively for the characteristic functions for Class A and B interference, to be used in (2.22b).

First, however, it is convenient to introduce the following normalizations:

Class A: $$\mathcal{E}_0 \equiv E_0/\sqrt{2\Omega_{2A}(1+\Gamma_A')} \quad ; \quad \mathcal{E} = E/\sqrt{2\Omega_{2A}(1+\Gamma_A')} \tag{3.1}$$

with

$$\Omega_{2A} \equiv A_{\infty,A}\langle \hat{B}^2_{0,A}\rangle/2 \quad ; \quad \Gamma_A' \equiv \sigma_G^2/\Omega_{2A}: \ \frac{\text{gauss intensity}}{\text{"impulsive" intensity}} \ . \tag{3.1a}$$

For Class B noise we use (5.14), viz.,

Class B: $$\mathcal{E}_0 \equiv \frac{E_0}{\sqrt{2\Omega_{2B}(1+\Gamma_B')}} \quad ; \quad \mathcal{E} \equiv \frac{E}{\sqrt{2\Omega_{2B}(1+\Gamma_B')}} \ , \tag{3.2}$$

where, cf. (5.14a), we have

$$\Omega_{2B} \equiv \left\{\frac{A_{\infty,B}\langle \hat{B}^2_{0,B}\rangle}{2}\right\} \quad ; \quad \Gamma_B' \equiv \sigma_G^2/\Omega_{2B} \tag{3.2a}$$

which are directly analogous to the corresponding parameters above for Class A noise. Then, writing

$$a_{AorB} \equiv \left\{2\Omega_2(1+\Gamma')\right\}^{-1/2}, \ A \text{ or } B \ , \tag{3.3}$$

we see that $\mathcal{E} = aE$, $\mathcal{E}_0 = aE_0$ in each case, and $\therefore r = a\lambda$ in (2.78), (2.90) and (2.93), so that the desired exceedance probabilities now have the generic form

$$P_1(\mathcal{E} > \mathcal{E}_o)_{A,B} = 1-\mathcal{E}_o\int_0^\infty J_1(\lambda\mathcal{E}_o)\hat{F}_1(ia\lambda)_{A,B}d\lambda, \tag{3.4}$$

cf. (2.22b), where, of course, the specific parameter values in the normalization factor $\underline{a}$ have different forms for the Class A and B interference.

3.1 Class A Interference:

Applying (3.1) - (3.3) to (2.78) and omitting the "correction terms" allows us to write the Class A c.f. in the following desired approximate form:

$$\hat{F}_1(ia\lambda)_A \doteq e^{-A_A}\sum_{m=0}^{\infty}\frac{A_A^m}{m!}e^{-\hat{\sigma}_{mA}^2 a^2\lambda^2/2} \quad ; \quad 2\hat{\sigma}_{mA}^2 \equiv (m/A_A+\Gamma_A')/(1+\Gamma_A') \tag{3.5}$$

cf. Eq. (3.17)* [Middleton, 1974], and we henceforth abbreviate $A_{\infty,A} = A_A$, etc. Applying (3.5) to (3.4) then gives us

$$P_1(\mathcal{E}>\mathcal{E}_o)_A \simeq 1 - \mathcal{E}_o e^{-A_A}\sum_{m=0}^{\infty}\frac{A_A^m}{m!}\int_0^\infty J_1(\mathcal{E}_o\lambda)e^{-\hat{\sigma}_{mA}^2 a^2\lambda^2/2}d\lambda, \tag{3.6}$$

which with the help of (2.25) and the relation ${}_1F_1(1;2;-x) = (1-e^{-x})/x$, [Eq. A.1-19b, Middleton, 1960], becomes**

$$\boxed{\begin{aligned} P_1(\mathcal{E}>\mathcal{E}_o)_A &\simeq 1-e^{-A_A}\sum_{m=0}^{\infty}\frac{A_A^m}{m!}\frac{\mathcal{E}_o^2}{2\hat{\sigma}_{mA}^2}\,{}_1F_1(1;2;-\mathcal{E}_o^2/2\hat{\sigma}_{mA}^2) && (3.7a) \\ &\simeq e^{-A_A}\sum_{m=0}^{\infty}\frac{A_A^m}{m!}e^{-\mathcal{E}_o^2/2\hat{\sigma}_{mA}^2} && (3.7b) \end{aligned}\ \Bigg\}\ \mathcal{E}_o \geq 0\,,}$$

------

* Note that $2\hat{\sigma}_{mA}^2$ here is equal to σ_{mA}^2, cf. Eq. (5.7), of Middleton (1974).

** This PD is properly normalized for $\langle\mathcal{E}^2\rangle_A = 1$, cf. remarks following Eq. (2.78).

for the desired approximation* of P_{1-A}. We observe at once that the Class A exceedance probability P_{1-A} is (primarily) a weighted sum of rayleigh probability distributions (P.D.'s), each with a variance which increases with order (m). Note from (3.7) that

$$P_1(\mathcal{E}\geq 0)_A = 1; \quad P_1(\mathcal{E}>\mathcal{E}_0\to\infty)_A = 0 ;$$

$$P_1(\mathcal{E}>\mathcal{E}_0 \doteq 0)_A \doteq 1-\left(e^{-A_A}\sum_{m=0}^{\infty}\frac{(1+\Gamma_A')\,A_A^m}{(\Gamma_A'+m/A_A)\,m!}\right)\mathcal{E}_0^2 \quad , \tag{3.8}$$

and since each term of (3.7b) is positive and the exponential less than unity, $0 \leq P_{1-A} \leq 1$, also with P_{1-A} monotonically decreasing as $\mathcal{E}_0 \to \infty$, all as required for a proper (exceedance) P.D. Furthermore, the expected "rayleigh"-form of the P.D. is exhibited in (3.8) for small thresholds, e.g.,

$$P_1(\mathcal{E}>\mathcal{E}_0)_A \doteq 1-B_A\mathcal{E}_0^2 \doteq e^{-B_A\mathcal{E}_0^2} \quad ; \quad B_A\mathcal{E}_0^2 \ll 1, \tag{3.9a}$$

where explicitly now

$$B_A \equiv (1+\Gamma_A')e^{-A_A}\sum_{m=0}^{\infty}\frac{A_A^{m+1}}{m!}\,\frac{1}{(m+A_A\Gamma_A')} \quad ,$$

which depends, of course, on the Impulsive Index A_A and on the intensity ratio Γ_A', cf. (3.1a).

---

* The correction terms (containing $\hat{C}_4, \hat{C}_6$, etc.) in the c.f., and hence in the P.D., may become important for extremely large $\mathcal{E}_0$ and very small values of the Index A_A, although present experimental results, and theory, indicate that the principal effects for large values of $\mathcal{E}_0$ are satisfactorily accounted for by the approximation (3.7). We reserve to a later study the investigation of these effects.

Figs. 3.1II and 3.2II, based on (3.7) show some typical distributions P_{1-A} vs. threshold $\mathcal{E}_0$, with Γ'_A and A_A respectively as parameters. As expected, these PD's are highly nonrayleigh* for the rarer "events", e.g. those which exceed the larger thresholds $\mathcal{E}_0$, while the rayleigh forms appear for the less rare events ($\mathcal{E}_0$ small), also as expected, cf. (3.9). Thus while the slope $(dP_1/d\mathcal{E}_0)$ is a constant -2 ($\equiv e^{-x^2}$) for the small amplitudes on these log vs. $\log^2$ probability scales, it is an (approximate) -1.2 for $\Gamma' = 10^{-4}$, $A_A = 10^{-1}$ for $\mathcal{E}_0 \to \infty$, i.e. a fall-off ($\equiv e^{-x^{(1.2)}}$ of P_{1-A} at large $\mathcal{E}_0$ somewhat faster than exponential, which latter is consistent with the required existence of all moments, cf. Sec. 5. Different values of A_A, Γ'_A lead to different limiting slopes as $\mathcal{E}_0 \to \infty$, but all are dominated by the exponential type of fall-off. In addition, as the relative size of the gaussian component increases (increasing Γ'_A) so does the threshold $\mathcal{E}_0$ rise, above which the nonrayleigh effects appear. Similarly, as the Impulsive Index A_A increases, i.e. the envelope distribution approaches eventually the limiting rayleigh form (2.57b) [with (2.53a)] as $A_A \to \infty$, the very steep "neck" of the curve becomes less extensive and shifts to the larger probabilities (lower $\mathcal{E}_0$), also as expected. In the limit $A_A \to \infty$ this "neck" disappears entirely and the straight-line (slope-2) rayleigh distribution appears, for all $\mathcal{E}_0$. [Development of these numerical results to a much more extensive and fine grid of parameter values is planned for a later Report.]

3.2 Class B Interference ($0 < \alpha < 2$):

Applying the normalizations (3.2)-(3.3) to (2.90), (2.93) now, for the two c.f.'s which approximate the Class B interference, we obtain explicitly**

---

* For envelopes E here, e.g., equivalently nongaussian for the corresponding instantaneous amplitudes X [Middleton, 1974].

** For compactness, we set $A_{\infty,B} \equiv A_B$, henceforth, cf. (3.5) et seq. above .

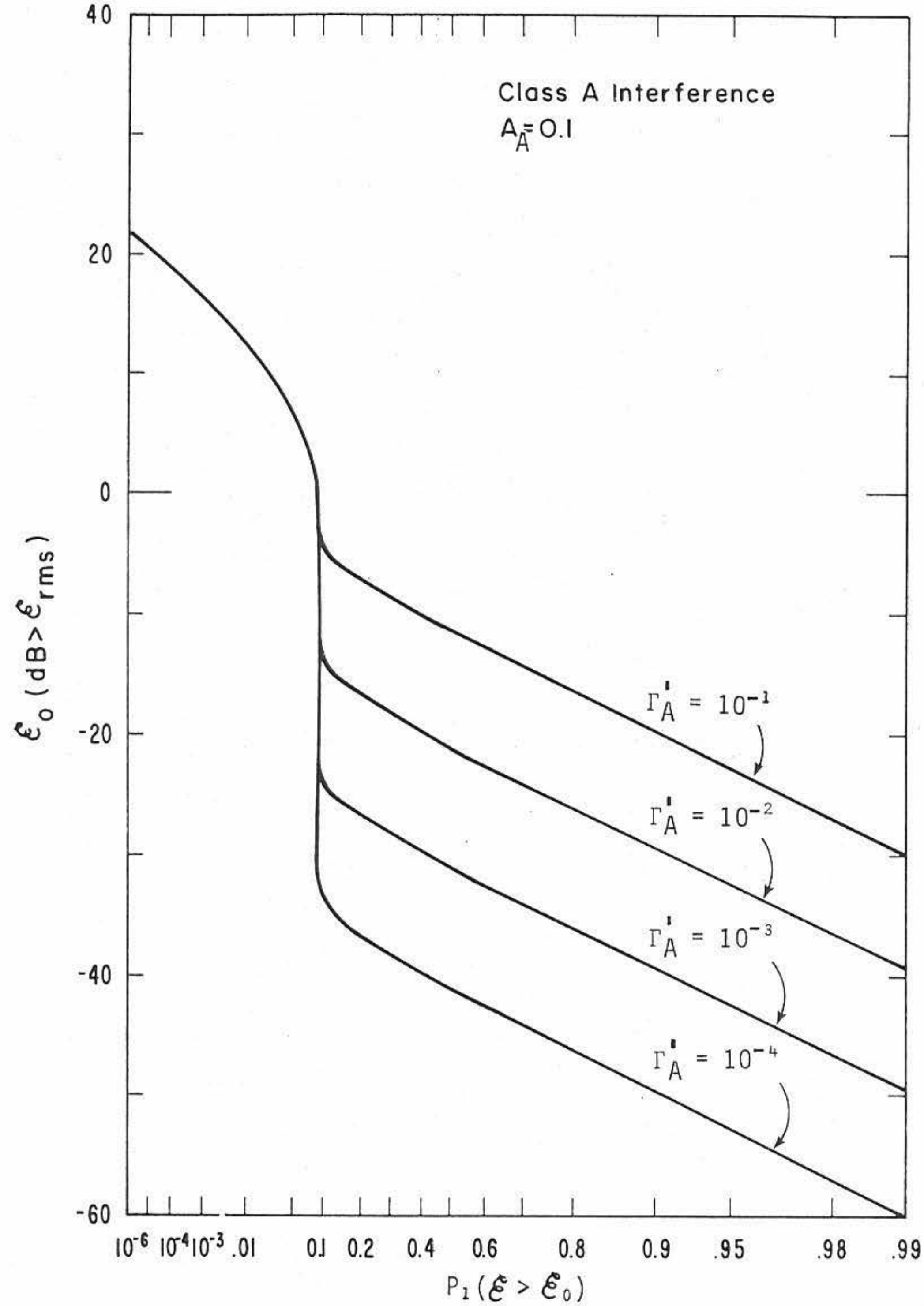

Figure 3.1 (II). The envelope distribution $[\mathrm{Prob}(\mathcal{E} > \mathcal{E}_0)]$ calculated for Class A interference for $A_A = 0.1$ and various Γ_A' from eq. (3.7b).

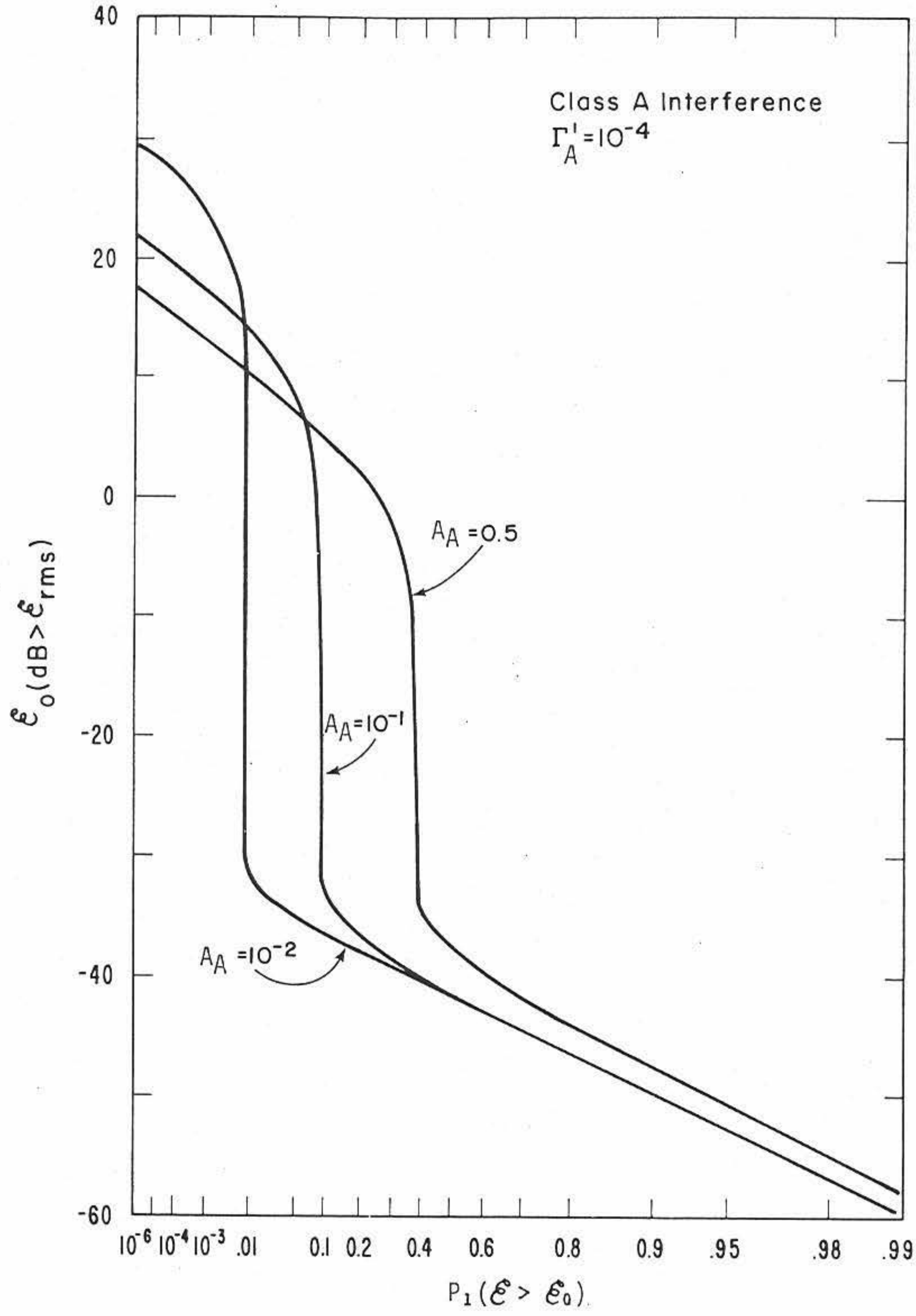

Figure 3.2 (II). The envelope distribution [Prob($\mathcal{E} > \mathcal{E}_0$)] calculated for Class A interference for $\Gamma_A' = 10^{-4}$ and various A_A from eq. (3.7b).

$$\hat{F}_1(ia\lambda)_{B-I} \doteq e^{-b_{1\alpha}A_B a^\alpha \lambda^\alpha - \Delta\sigma_G^2 a^2\lambda^2/2} \quad , \tag{3.10a}$$

$$\hat{F}_1(ia\lambda)_{B-II} \doteq e^{-A_B}\cdot \exp[A_B e^{-b_{2\alpha}a^2\lambda^2/2} - \sigma_G^2 a^2\lambda^2/2]. \tag{3.10b}$$

These, in turn are applied to (3.4) to give us $P_{1-I,II}$ respectively. Starting with (3.10a), we expand the exponential in λ^α and use (2.55) to get

$$P_1(\mathcal{E}>\mathcal{E}_0)_{B-I} \simeq 1-\mathcal{E}_0 \sum_{n=0}^{\infty} \frac{(b_{1\alpha}A_B a^\alpha)^n}{n!} (-1)^n \int_0^\infty \lambda^{n\alpha} J_1(\lambda\mathcal{E}_0) e^{-\Delta\sigma_G^2 a^2\lambda^2/2} d\lambda \tag{3.11a}$$

$$\equiv \hat{P}_1(\hat{\mathcal{E}}>\hat{\mathcal{E}}_0) \simeq 1-\hat{\mathcal{E}}_0^2 \sum_{n=0}^{\infty} \frac{(-1)^n \hat{A}_\alpha^n}{n!} \Gamma(1+\frac{\alpha n}{2})\, {}_1F_1(1+\frac{\alpha n}{2}; 2; -\hat{\mathcal{E}}_0^2) \quad , \tag{3.11b}$$

with

$$\hat{\mathcal{E}}_0 \equiv (\mathcal{E}_0 N_I)/2G_B; \ \hat{A}_\alpha \equiv A_\alpha/2^\alpha G_B^\alpha \ , \qquad [\mathcal{E}_0 \to \mathcal{E}_0 N_I \text{ in } (3.11a)], \tag{3.11c}$$

$$A_\alpha \equiv 2^\alpha b_{1\alpha} a^\alpha A_B = 2^\alpha b_{1\alpha} A_B/[2\Omega_{2B}(1+\Gamma_B')]^{\alpha/2} = \frac{2\Gamma(1-\alpha/2)}{\Gamma(1+\ /2)} A_B \left\langle \left(\frac{\hat{B}_{o,B}}{\sqrt{2\Omega_{2B}(1+\Gamma_B')}} \right)^\alpha \right\rangle \tag{3.12a}$$

and

$$G_B^2 = \frac{1}{4}(1+\Gamma_B')^{-1}(\frac{4-\alpha}{2-\alpha} + \Gamma_B') \quad , \tag{3.12b}$$

cf. (2.88a,b,c), (3.3a), where N_I is a <u>scaling</u> factor which scales $P_{1-(B-I)}$, $w_{1-(B-I)}$ to insure that $\langle \mathcal{E}^2 \rangle_B = 1$, cf. (2.94) ff, where $a_B^2 \Delta\sigma_G^2 \equiv 2G_B^2$. The quantity $\hat{A}_\alpha$ is the <u>Effective Class B Impulsive Index</u>, which is proportional to the Impulsive Index A_B, for this Class B interference. In addition, it depends spatially on the spatially sensitive parameter, α, and on the relative gauss component Γ_B', (3.2a).

With the help of Kummer's transformation [Middleton, 1960, Section A.1.2, p. 1073, Eq. A.1-17] we can write (3.11b) alternatively as

$$\hat{P}_1(\hat{\mathcal{E}}>\hat{\mathcal{E}}_0)_{B-I} \simeq e^{-\hat{\mathcal{E}}_0^2} \left\{ 1-\hat{\mathcal{E}}_0^2 \sum_{n=1}^{\infty} \frac{(-1)^n \hat{A}_\alpha^n}{n!} \Gamma(1+\frac{\alpha n}{2})\, {}_1F_1(1-\frac{\alpha n}{2}; 2; \hat{\mathcal{E}}_0^2) \right\} , \tag{3.13}$$

where we have used Eq. (A.1-19b) [Middleton, 1960]. For large $\mathcal{E}_0$ we obtain formally, with the help of the asymptotic relation [Middleton, 1960, (A.1-16b), p. 1073],

$$
{}_1F_1(\alpha;\beta;-x) \simeq \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\, x^{-\alpha} \left[1+\frac{\alpha(\alpha-\beta+1)}{1!x}+\frac{\alpha(\alpha+1)(\alpha-\beta+1)(\alpha-\beta+2)}{2!x^2}+\ldots\right], \tag{3.14}
$$

the following expression for P_1:

$$
\hat{P}_1(\hat{\mathcal{E}}>\hat{\mathcal{E}}_0)_{B-I} \simeq \sum_{n=1}^{\infty} \frac{\hat{A}_\alpha^n(-1)^{n+1}}{n!} \frac{\Gamma(1+\frac{\alpha n}{2})}{\Gamma(1-\frac{\alpha n}{2})}\, \hat{\mathcal{E}}_0^{-n\alpha} \left[1+\frac{(1+\alpha n/2)(\alpha n)}{2\hat{\mathcal{E}}_0^2}+\ldots\right],
$$

$$
\mathcal{E}_0^2 \gg 1. \tag{3.15}
$$

This shows that $\lim_{\mathcal{E}_0\to\infty} P_{1-I} \to O(\mathcal{E}_0^{-\alpha}) \to 0$. However, as explained in A, Section 2.7, for $\mathcal{E}_0$ greater than some (large) value $\mathcal{E}_B$, which is determined from Eq. (3.19e) below, we must use the second form of c.f., (3.10b). Figs.3.3II,3.4II here are based on (3.11b), (3.15), and are valid representations, provided $\mathcal{E}_0$ is not too large, e.g. $\mathcal{E}_0 \leq \mathcal{E}_B$.

For the "rare events", or large $\mathcal{E}_0$, we apply (3.10b) to (3.4), as discussed earlier (cf. A, Sec. 2.7), to obtain

$$
P_1(\mathcal{E}>\mathcal{E}_0)_{V-II} \simeq 1-\mathcal{E}_0 e^{-A_B} \sum_{m=0}^{\infty} \frac{A_B^m}{m!} \int_0^\infty J_1(\mathcal{E}_0\lambda)\, e^{-\hat{\sigma}_{mB}^2 a^2\lambda^2/2}\, d\lambda, \tag{3.16}
$$

with

$$
2\hat{\sigma}_{mB}^2 = 2(mb_{2\alpha}+\sigma_G^2)a_B^2 = \left(\frac{m}{\hat{A}_B}+\Gamma_B'\right)/(1+\Gamma_B')\ , \quad \hat{A}_B \equiv A_B\left(\frac{2-\alpha}{4-\alpha}\right) \tag{3.16a}
$$

from (3.2a),(2.88c), cf.(3.5), (3.6): thus $\hat{\sigma}_{mB}$ has the same <u>form</u> as $\hat{\sigma}_{mA}$, (3.6). Accordingly, we may use the result (3.7b), rewriting it here for this large-magnitude approximation for Class B noise, as*

$$
\boxed{P_1(\mathcal{E}>\mathcal{E}_0)_{B-II} \simeq \frac{e^{-A_B}}{4G_B^2} \sum_{m=0}^{\infty} \frac{A_B^m}{m!}\, e^{-\mathcal{E}_0^2/2\hat{\sigma}_{mB}^2}\ , \quad (\mathcal{E}_0 > \mathcal{E}_B)\ .} \tag{3.17}
$$

------

* This PD is now properly normalized [remarks after (2.94) and], cf. (i), (3.17a,b), and discussion following.

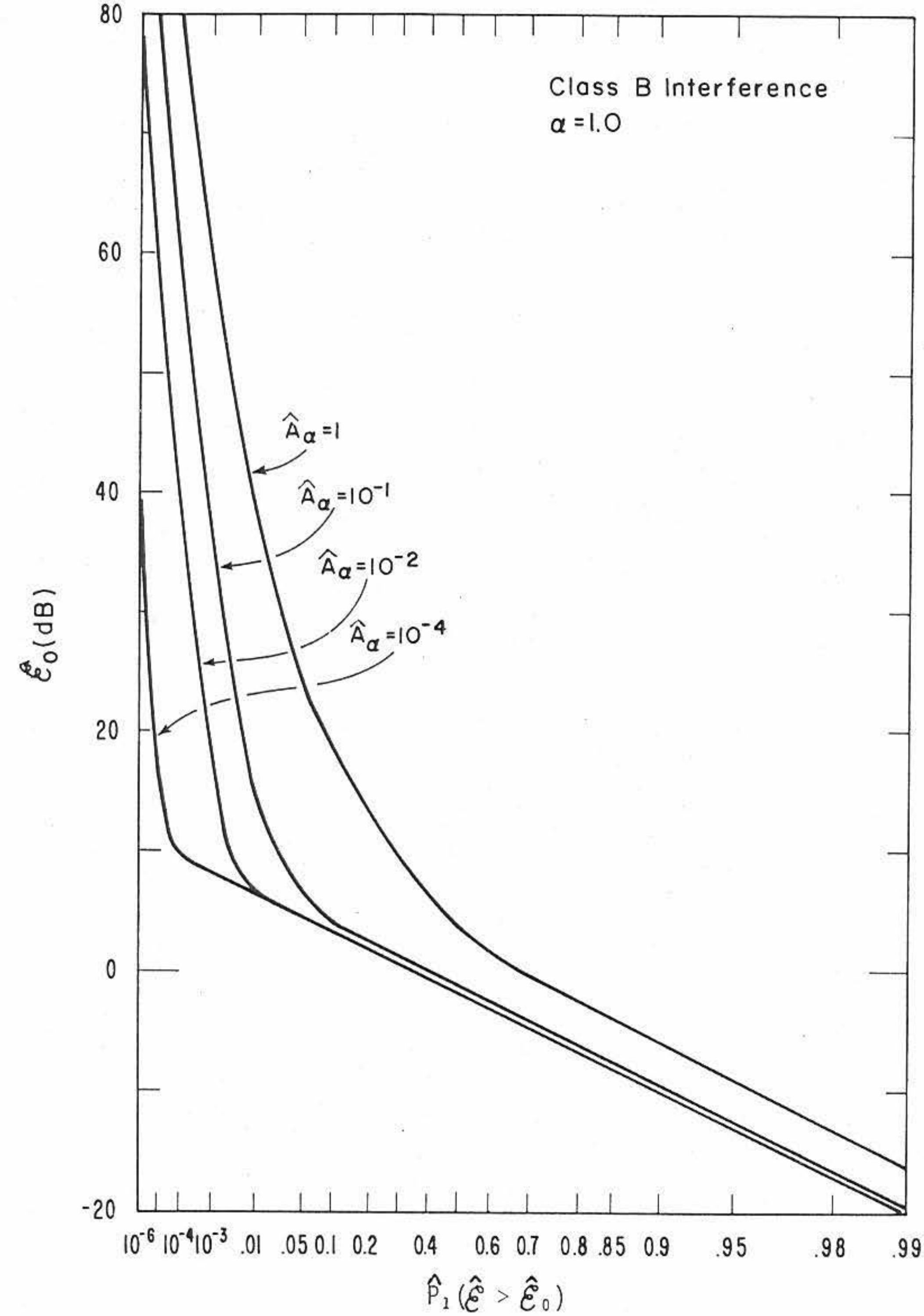

Figure 3.3 (II). The envelope distribution $[\mathrm{Prob}(\hat{\mathcal{E}} > \hat{\mathcal{E}}_0)]$ calculated for Class B interference for $\alpha = 1.0$ for various $\hat{A}_\alpha$ from eqs. (3.11b, 3.15).

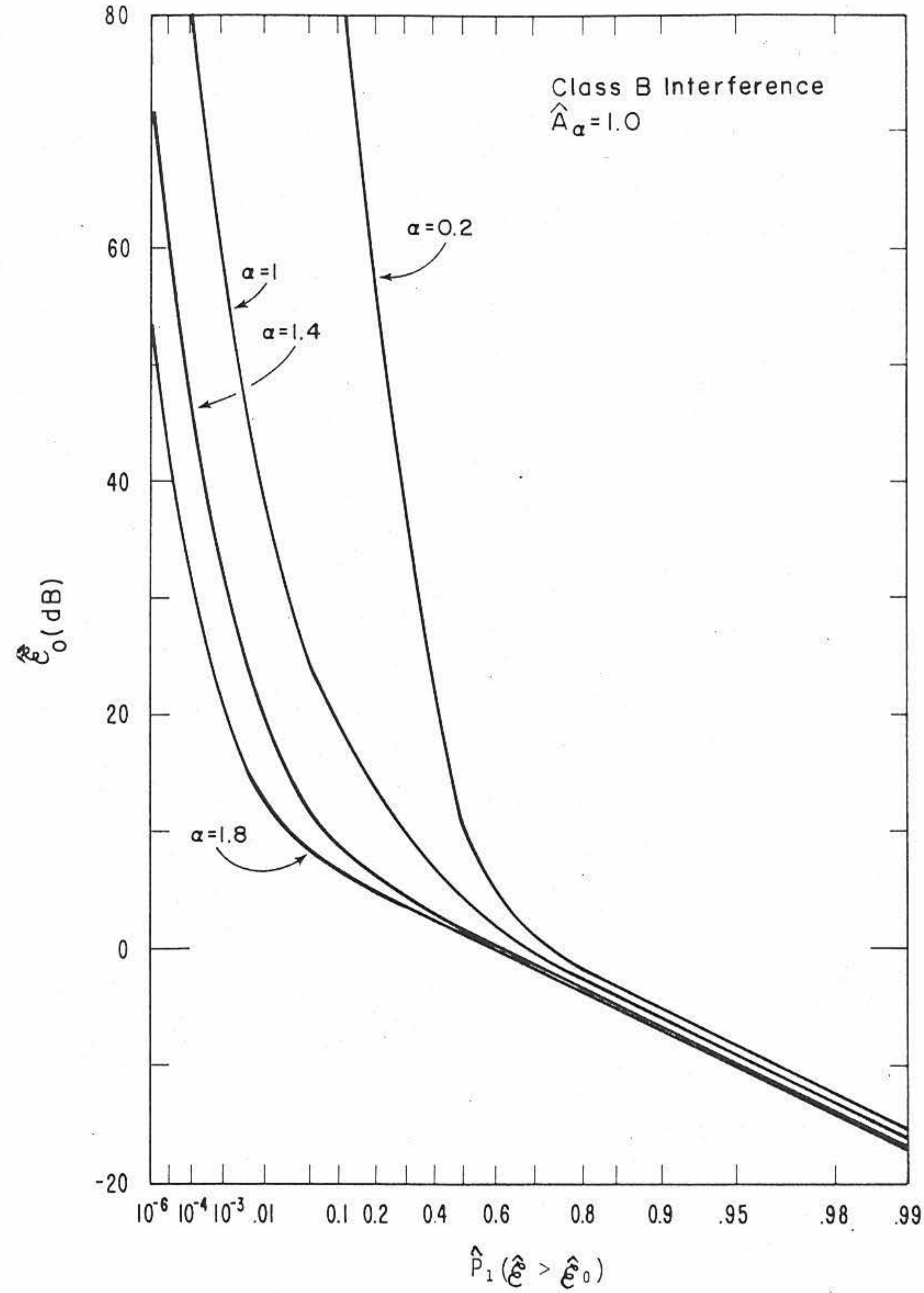

Figure 3.4 (II). The envelope distribution $[\mathrm{Prob}(\hat{\mathcal{E}} > \hat{\mathcal{E}}_0)]$ calculated for Class B interference for $\hat{A}_\alpha = 1.0$ for various α from eqs. (3.11b, 3.15).

[Figures 3.1II, 3.2II for Class A interference illustrate the character of (3.17), which, of course, is only applicable here, Class B, for the larger values of $\mathcal{E}_0$ ($>\mathcal{E}_B$).]

A. The Composite Approximation:

The problem with the approximating results for P_{1-B} in the Case B model, cf. (3.11b) and (3.17), is that these forms, stemming as they do from approximate c.f.'s [cf. (3.10a,b)], are not properly scaled, or "normalized", in the sense that each approximating form, P_{1-I}, P_{1-II}, does not yield the correct mean square value of $\langle\mathcal{E}^2\rangle_B = 1$ or $\langle E^2\rangle_B = 2\Omega_{2B}(1+\Gamma_B')$, cf. (5.14) with (3.2), and the remarks following Eq. (2.94) above. The approximation P_{1-I}, and its associated pdf, w_{1-I}, (4.3), in fact, do not possess a finite mean square on $(0,\infty)$, cf. Sec. (5.3)ff., while P_{1-II}, the "Type A" form and its pdf., w_{1-II}, (4.4), yields $\langle\mathcal{E}^2\rangle_{B-II} \neq 1$.

Accordingly, since the precise mean square is finite and is known to be $\langle\mathcal{E}^2\rangle_B = 1$, by calculation from the exact c.f. [cf. (5.10a), and Section 5.2-B], we must suitably scale (or "normalize") w_{1-I}, w_{1-II} (4.5) so that $\langle\mathcal{E}^2\rangle_B$, cf. (5.6c), exists and is equal to unity. This is done as follows:

(i). Let us consider first w_{1-II}, (4.4), and calculate $\langle\mathcal{E}^2\rangle_{II}$ on $(0<\mathcal{E}<\infty)$ according to (5.1). The result is easily seen to be

$$\langle\mathcal{E}^2\rangle_{B-II} = e^{-A_B} \sum_{m=0}^{\infty} \frac{(m/\hat{A}_B+\Gamma_B')}{(1+\Gamma_B')m!} A_B^m = \frac{\frac{4-\alpha}{2-\alpha}+\Gamma_B'}{1+\Gamma_B'} = 4G_B^2(\neq 1), \tag{3.17a}$$

where G_B^2 is given by (3.12b), so that here we require the normalization factor $N_{II}^2 = (4G_B^2)^{-1}$, e.g.

$$w_1(\mathcal{E})_{B-II\text{-norm}} = \frac{1}{4G_B^2} w_1(\mathcal{E})_{B-II} = N_{II}^2 w_1(\mathcal{E})_{B-II}. \tag{3.17b}$$

(Henceforth in the text we write $w_1(\mathcal{E})_{B-II}, P_{1-II}$ in normalized form, which are then used for anlytical and numerical calculations in the remaining sections of Part II here.)

(ii). The case of w_{1-I}, (4.3), requires a different approach, since $\langle\mathcal{E}^2\rangle_{B-I}$ on $(0<\mathcal{E}<\infty)$ becomes infinite $(0<\alpha<2)$, cf. Section 5.3: [$\langle\mathcal{E}^2\rangle_{B-II}$ on

$(0<\mathcal{E}\leq\mathcal{E}_B<\infty)$, of course, is finite, cf. (5.6c)]. Here we need to scale $\mathcal{E}_0$ according to (3.11b) above: $\underline{\mathcal{E}_0 \to \mathcal{E}_0 N_I}$ (and $\therefore \hat{\mathcal{E}}_0 = (\mathcal{E}_0 N_I)/2G_B$). The rationale for this is the observation that P_{1-I} (and w_{1-I}) must have the same values in the rayleigh region ($\mathcal{E}_0^2<<1$), where $P_{1-I} \sim 0.9$, or 0.99, etc., as does the precise distribution, P_{1-B}, based on the (intractable but) exact c.f. (2.87), hence (3.11b). The scaling factor, N_I, is to be determined by fitting the two approximate forms P_{1-I}, P_{1-II} together by the procedure outlined below, which is based on the canonical properties of the Class B model generally. Note, finally, that the "Class A" form (II) is coupled to the Class B form (I) through the Class B parameter α, and vice versa through the "Class A" parameter Γ_B', appearing in G_B, common to both approximations I,II.

To combine the suitably scaled P_{1-I} and normalized P_{1-II} to form the composite approximation for Class B interference which is valid for all $\mathcal{E}_0 \geq 0$ we now use the following desired properties of $P_{1\text{-composite}}$, which is sketched in Fig. 3.5II:

(i). $P_{1-I} = P_{1-II}$ in the rayleigh region, e.g. $0 \leq \mathcal{E}_0$ small. Equality of the two approximations in the rayleigh region is required, since both must represent the same (small) amplitude behaviour, characteristic of all these PD's.

(ii). $\dfrac{dP_{1-I}}{d\mathcal{E}_0} = \dfrac{dP_{1-II}}{d\mathcal{E}_0}$ in the rayleigh region.

(iii). $P_{1-I} = P_{1-II}$ at the "bendover" or junction point $\mathcal{E}_B$ of the two approximations, cf. Fig.(3.5)II. This point, $\mathcal{E}_B$, is empirically determined from the data, e.g. from the experimental APD or exceedance probability curve $P_1(\mathcal{E} > \mathcal{E}_0)_{exp.}$, as described below, cf. (vi).

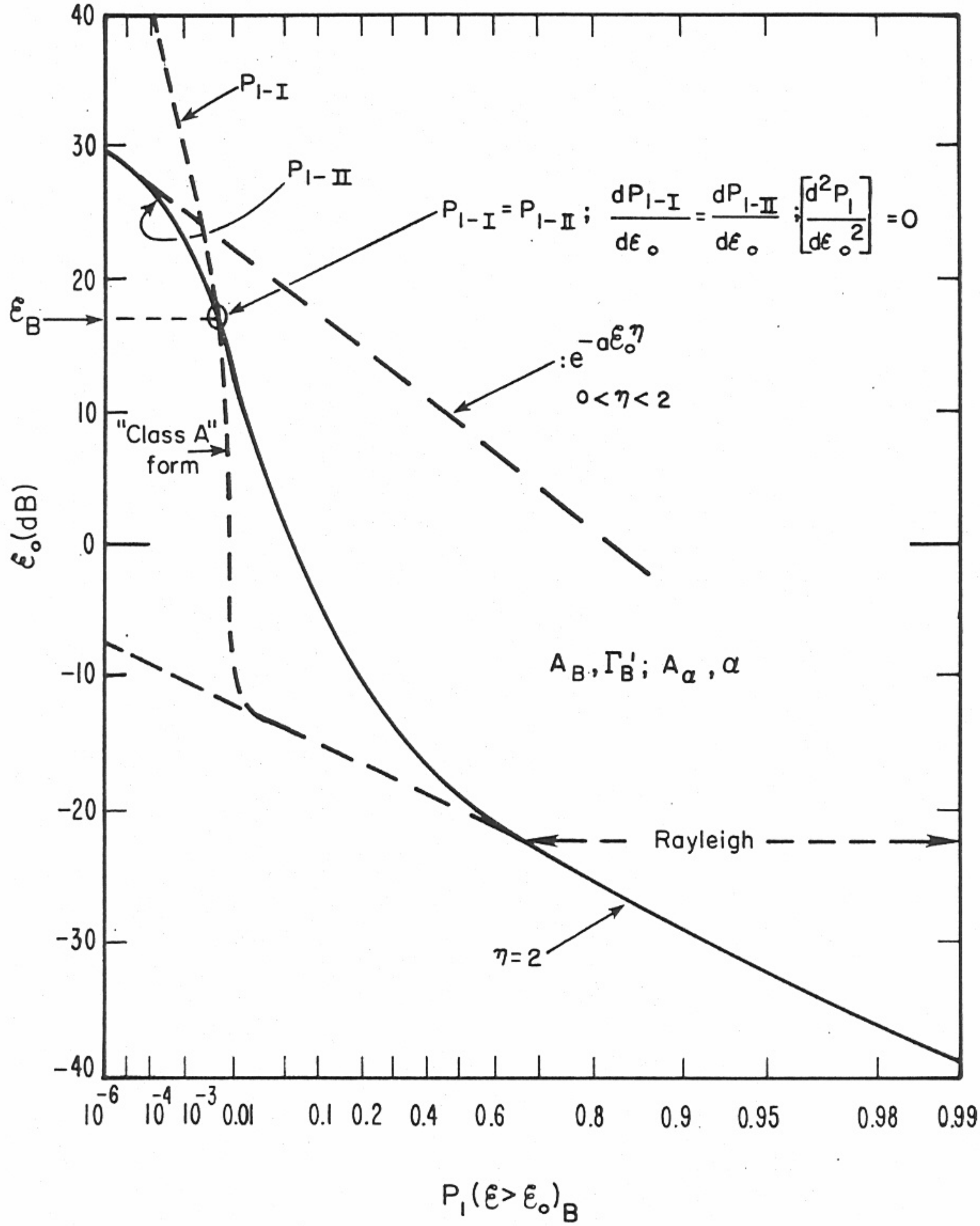

Figure 3.5 (II). Schema of P_{1-B}, eq. (3.20), obtained by joining the two approximating forms (3.11b, 3.17).

(iv). $\left(\frac{dP_{1-I}}{d\mathcal{E}} = \frac{dP_{1-II}}{d\mathcal{E}}\right)_{\mathcal{E}_B}$: the (finite) slopes of the approximating $P_{1-I,II}$ are equal at $\mathcal{E}_B$: this insures a common tangent, i.e., a smooth fit; moreover, we have

(v). $\left(\frac{d^2P_{1-I}}{d\mathcal{E}^2} = \frac{d^2P_{1-II}}{d\mathcal{E}^2}\right)_{\mathcal{E}_B}$ ($\neq 0$): this follows as a consequence of (iv), and the continuity of the derivative at $\mathcal{E}_B$, insuring that the associated pdf's are continuous at the joining point $\mathcal{E}_B$. However, note that $\mathcal{E}_B$ is not usually a point of inflexion of $P_{1-I,II}$.

(vi). $\mathcal{E}_B$: this is the point of inflexion ($d^2P_1/d\mathcal{E}_B^2 = 0$) of the actual P_1, and is determined as such (by inspection, usually), of the empirical exceedance probability $P_{1-exp.}$, cf. Fig. (3.5)II. (3.18)

Accordingly, from (3.11b), (3.15), (3.17) we have explicitly for (i)-(v) above:

$$\text{(i).}\quad \frac{\mathcal{E}_0^2 N_I^2}{4G_B^2}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{A_\alpha}{2^\alpha G_B^\alpha}\right)^n}{n!}\Gamma\left(1+\frac{\alpha n}{2}\right) \doteq \frac{\mathcal{E}_0^2 e^{-A_B}}{4G_B^2}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\left(2\hat{\sigma}_{mB}^2\right)^{-1} ; \tag{3.19a}$$

[(ii). [Same as (3.19a), without the $2\mathcal{E}_0$ factors common to both members of the equation; however, (ii) is here implied by the <u>form</u> of (i) and does not provide new information.] (3.19b)

$$\text{(iii).}\quad \frac{A_\alpha\Gamma(1+\alpha/2)}{2^\alpha G_B^\alpha\Gamma(1-\alpha/2)}\left(\frac{\mathcal{E}_B N_I}{2G_B}\right)^{-\alpha} 1+O_{(iii)}([\mathcal{E}_B N_I]^{-\alpha};\mathcal{E}_B^2 N_I^2)] = \frac{e^{-A_B}}{4G_B^2}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}e^{-\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2};$$

$$\text{(iv).}\quad \frac{A_\alpha\alpha\Gamma(1+\alpha/2)}{2^\alpha G_B^\alpha\Gamma(1-\alpha/2)}\left(\frac{\mathcal{E}_B N_I}{2G_B}\right)^{-\alpha-1}[1+O_{(iv)}([\mathcal{E}_B N_I]^{-\alpha}[\mathcal{E}_B N_I]^3)] \simeq \frac{\mathcal{E}_B e^{-A_B}}{4G_B^2}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}e^{-\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2}; \tag{3.19d}$$

(v),(vi): $\mathcal{E}_B$ cannot be determined analytically from either approximating form $P_{1-I,II}$. It must be established as a point of inflexion from the <u>empirical</u> PD, as noted above. (3.19e)

[In using (iii), (iv), we may need at least the next set of "correction" terms in the asymptotic developments of P_{1-I}, $dP_{1-I}/d\mathcal{E}_B$.] We note that (given $\mathcal{E}_B$) these three relations [(i),(iii),(iv)] are sufficient to determine in principle, any three of the six parameters $N_I,\alpha,A_\alpha,A_B,\Omega_{2B},\Gamma_B'$ (cf. (3.16a)), when the other three are specified. Later, in Section 6, we shall show how (3.18), (3.19) may be extended to permit us to obtain, from the experimental exceedance probability $P_1(\mathcal{E}>\mathcal{E}_0)_{xpt}$, the six parameters $N_I,\alpha,A_\alpha,A_B,\Omega_{2B},\Gamma_B'$ (or, more fundamentally, $[\alpha,A_B,\langle\hat{B}^\alpha_{0,B}\rangle,\Omega_{2B},\Gamma_B']$, cf. (3.12)).

For the illustrative calculations of Figs. 3.6II, 3.7II, it is convenient to preset $\mathcal{E}_B;N_I,\alpha,A_\alpha$, and determine $A_B,\Gamma_B',\Omega_{2B}$ from (i),(iii),iiv). Other possibilities are: Fix $(\mathcal{E}_B;N_I,\alpha,\Gamma_B')$, determine $(A_\alpha,\Omega_{2B},A_B)$; fix $(\mathcal{E}_B;A_\alpha,\alpha,\Gamma_B')$, determine (N_I,A_B,Ω_{2B}); fix $(\mathcal{E}_B;A_B,\Gamma_B',\Omega_{2B})$, determine (N_I,α,A_α); fix $(\mathcal{E}_B;N_I,A_B,\Gamma_B')$, determine $(A_\alpha,\alpha,\Omega_{2B})$ etc. In any case, we have now

$$P_{1-B} = P_{1-I},\ 0 \le \mathcal{E}_0 \le \mathcal{E}_B;\ = P_{1-II},\ \mathcal{E}_0 \ge \mathcal{E}_B, \tag{3.20}$$

with P_{1-I}, P_{1-II} given respectively by (3.11b) and (3.17). The curves of Figs. 3.6II, 3.7II are equivalent to Figs. 3.1.1, 3.1.2 of Furutsu and Ishida [1960], with $(\nu/\alpha)_{F+I} \to A_B$, $(a_0/\sigma)_{F+I} \to (\Gamma_B')^{-1}$, and $(R/\sigma)_{F+I} \to \mathcal{E}_0$ and exhibit the same kind of "elbow" in the transition region from the rayleigh behaviour (for $\mathcal{E}_0^2<<1$), with a bend-over to a constant slope ($P_1 \sim e^{-a\mathcal{E}_0^\eta}$, $\eta>0$), as for Class A noise, when $\mathcal{E}_0 \to \infty$, cf. Figs.3.1II,3.2II, 3.5II.

B. Remarks on Hall-Type Models:

Finally, we observe that a Hall model [Hall, 1966] may be obtained formally from the P_{1-I} form for the rayleigh and intermediate region ($0\le\mathcal{E}_0\le\mathcal{E}_B$), provided we neglect the gaussian contributions (e.g. $\Delta\sigma_G^2 \to 0$), so that the c.f. (2.89) now applies. From (2.89) in (3.4) we accordingly

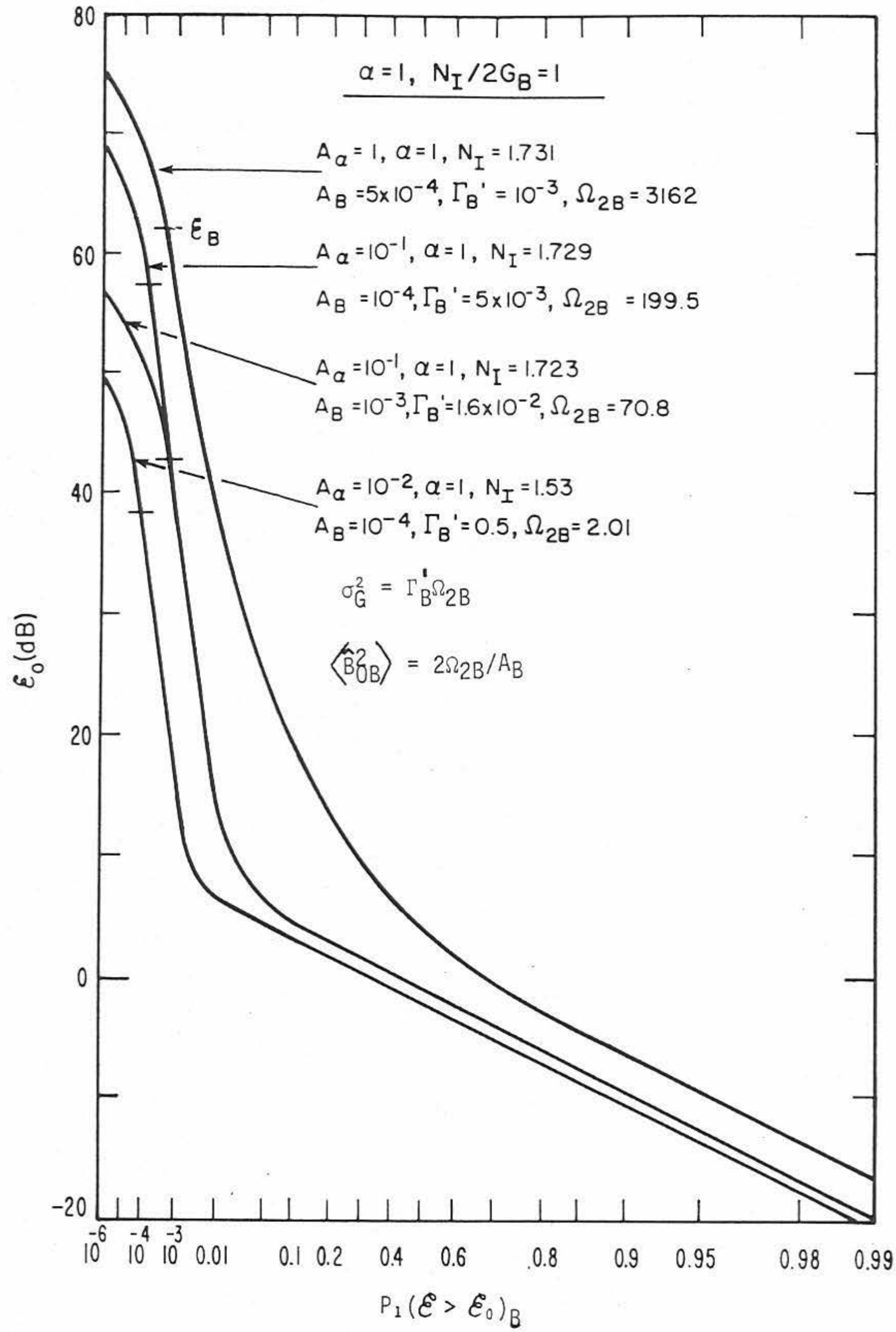

Figure 3.6 (II). The (complete) envelope distribution $P_1(\mathcal{E} > \mathcal{E}_0)_B$, eq. (3.20), calculated for Class B interference for various A_α, given α eqs. (3.11, 3.17, 3.19, 6.9, 6.10) .

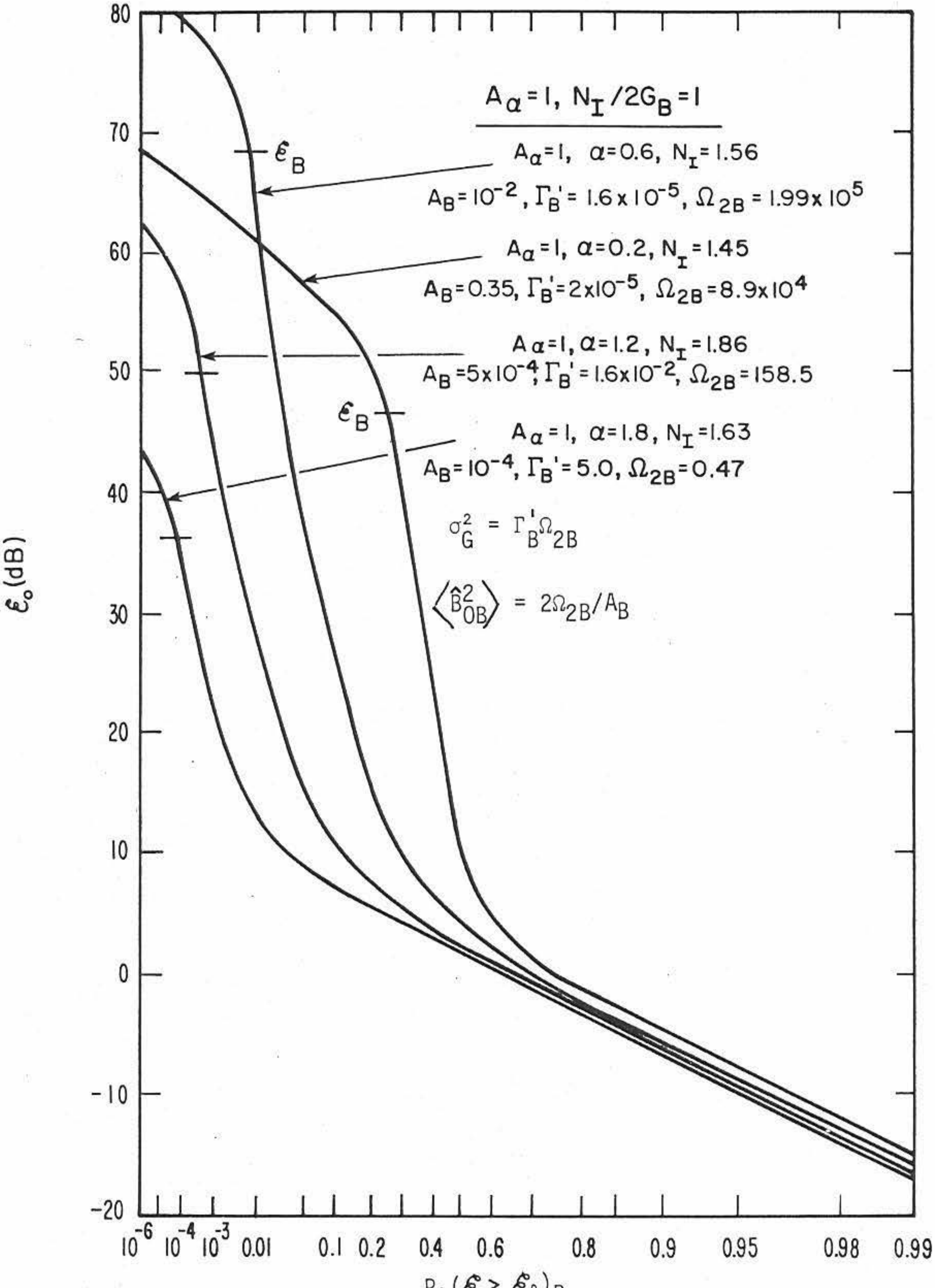

Figure 3.7 (II). The (complete) envelope distribution $P_1(\mathcal{E} > \mathcal{E}_0)_B$, eq. (3.20), calculated for Class B interference for various α, given A_α (eqs. 3.11, 3.17, 3.19, 6.9, 6.10).

obtain*

$$P_1(\mathcal{E} > \mathcal{E}_0)'_{B-I} \simeq 1-\mathcal{E}_0\int_0^\infty J_1(\mathcal{E}_0\lambda)e^{-b_{1\alpha}A_B a^\alpha \lambda^\alpha} d\lambda. \tag{3.21}$$

This integral may be evaluated in several ways, convenient for small or large $\mathcal{E}_0$. We start with the case convenient for the small values, and employ the following transformations:

$$\left.\begin{aligned} B_\alpha \equiv b_{1\alpha}A_B a^\alpha (=A_\alpha \mathcal{E}^{-\alpha}); \quad B_\alpha \lambda^\alpha = z \; ; \; \therefore \lambda = (z/B_\alpha)^{1/\alpha} \\ d\lambda = dz\ z^{(1-\alpha)/\alpha}/\alpha B_\alpha^{1/\alpha} \end{aligned}\right\}, \tag{3.22}$$

so that (3.21) becomes now

$$P_1(\mathcal{E}>\mathcal{E}_0)'_{B-I} \simeq 1-\frac{\mathcal{E}_0^*}{\alpha}\int_0^\infty J_1(\mathcal{E}_0^* z^{1/\alpha})z^{\frac{1-\alpha}{\alpha}} e^{-z}dz \; , \; \mathcal{E}_0^* \equiv \mathcal{E}_0/B_\alpha^{1/\alpha} \; . \tag{3.23}$$

Next, let us use the Barnes integral representation of J_1:

$$J_1(\mathcal{E}_0^* z^{1/\alpha}) = \int_\Gamma \frac{\Gamma(-s)\mathcal{E}_0^{*2s+1}}{\Gamma(s+2)2^{2s+1}} z^{(2s+1)/\alpha} \frac{ds}{2\pi i} \quad , \tag{3.24}$$

cf. Eq. (13.106) and Fig. 13.22 of [Middleton, 1960], where Γ is the contour $(-\infty i+c,\ i\infty+c)$, with $c(<0)$ chosen so that the integral over z in (3.23) is convergent at $z = 0$, e.g.

$$\int_0^\infty z^{(2+2s-\alpha)/\alpha}e^{-z}dz = \Gamma\left(\frac{2+2s}{\alpha}\right) \; , \; \mathrm{Re}(s) > -1, \quad \therefore -1 < c < 0. \tag{3.25}$$

---

* Equation (3.21) was obtained earlier by Giordano [1970, Eq. 3.66 therein; Giordano and Haber, 1972, Eq. 24] but was not analytically evaluated. Moreover, $P_{1_{B-I}}$, here (as well as the earlier forms [Giordano, 1970], etc.) is not scaled, e.g., $\langle \mathcal{E}^2 \rangle_{B-I} \neq 1$, as discussed at the beginning of A. above. See comments at end of B here.

Thus we find that (3.23) becomes

$$P_1(\mathcal{E}>\mathcal{E}_0)'_{B-I} \simeq 1-\frac{\mathcal{E}_0^*}{\alpha}\int_\Gamma \frac{\Gamma(-s)}{\Gamma(s+2)}\,\Gamma\left(\frac{2+2s}{\alpha}\right)\left(\frac{\mathcal{E}_0^*}{2}\right)^{2s+1}\frac{ds}{2\pi i} \tag{3.26a}$$

$$\simeq 1-\frac{2\mathcal{E}_0^2}{\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma\left(\frac{2+2n}{\alpha}\right)\mathcal{E}_0^{2n}}{n!(n+1)!A_\alpha^{(2+2n)/\alpha}}\quad , \tag{3.26b}$$

this last on evaluating (3.26a) at the simple poles $s = n = 0,1,2,...$ and using $B_\alpha = A_\alpha 2^{-\alpha}$, cf. (3.22). Equation (3.26b) exhibits the characteristic rayleigh form ($\sim\mathcal{E}_0^2$), when $\mathcal{E}_0^2 \ll 1$, as expected.

Next, for large values of $\mathcal{E}_0$ (or small values of A_α), we return to (3.21) and use the Barnes integral representation for $\exp(-B_\alpha\lambda^\alpha)$, viz:

$$e^{-B_\alpha\lambda^\alpha} = \int_\Gamma \Gamma(-s)B_\alpha^s\lambda^{\alpha s}\frac{ds}{2\pi i}\quad , \tag{3.27}$$

to reëxpress (3.21) as

$$\left.\begin{aligned} P_1(\mathcal{E}>\mathcal{E}_0)'_{B-I} \simeq 1-\int_\Gamma \mathcal{E}_0^{*-\alpha s}\Gamma(-s)\frac{ds}{2\pi i}\int_0^\infty z^{\alpha s}J_1(z)dz, \quad & -2 < \mathrm{Re}(\alpha s) < 0, \\ & \mathrm{Re}(s) > -1. \end{aligned}\right\} \tag{3.28}$$

To evaluate the z-integral we use [Watson, 1944,p. 391]:

$$\int_0^\infty J_\nu(t)t^{\mu-\nu-1}dt = \Gamma(\mu/2)/\Gamma(\nu-\mu/2+1)2^{\nu-\mu+1},\ \mathrm{Re}(\mu)<\mathrm{Re}(\nu+3/2) \tag{3.29}$$

with $\nu=1, \mu = s+2$ here, and $\therefore 0<\mathrm{Re}(\alpha s+2)<2<\mathrm{Re}\ 5/2$, as required. Equation (3.28) becomes

$$P_1(\mathcal{E}>\mathcal{E}_0)'_{B-I} \simeq 1-\int_\Gamma \frac{\Gamma(-s)\Gamma(1+\frac{\alpha s}{2})}{\Gamma(1-\frac{\alpha s}{2})}2^{\alpha s}\mathcal{E}_0^{*-\alpha s}\frac{ds}{2\pi i} \tag{3.30a}$$

$$\simeq \sum_{n=1}\frac{\Gamma(1+\frac{\alpha s}{2})\ (-1)^{n+1}A_\alpha^n}{\Gamma(1-\frac{\alpha n}{2})n!\,\mathcal{E}_0^{\alpha n}}\quad , \tag{3.30b}$$

which shows how this approximation behaves as $\mathcal{E}_o \to \infty$, viz. $O(\mathcal{E}_o^{-\alpha})$, e.g.:

$$P_1(\mathcal{E}>\mathcal{E}_o)'_{B-I} \simeq \frac{\Gamma(1+\alpha/2)A_\alpha}{\Gamma(1-\alpha/2)\mathcal{E}_o^\alpha} - \frac{\Gamma(1+\alpha)}{2\Gamma(1-\alpha)} \frac{A_\alpha^2}{\mathcal{E}_o^{2\alpha}} + O(\mathcal{E}_o^{-3\alpha}) \ , \ 0 < \alpha < 2. \tag{3.31}$$

Now in the special case $\alpha = 1$, we may sum the series (3.26b) or (3.30b). Choosing (3.26b) we get

$$P_1(\mathcal{E}>\mathcal{E}_o)'_{B-I} \simeq 1-2\mathcal{E}_o^2 \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(2+2n)\mathcal{E}_o^{2n}}{n!(n+1)!A_1^{2+2n}} \ , \tag{3.32a}$$

and since $(n+1)! = (2)_n$, $\Gamma(2+2n) = 2^{2n}(1)_n(3/2)_n$ [Middleton, 1960, (A.1-46b), p. 1078], we find that

$$P_1(\mathcal{E}>\mathcal{E}_o)'_{B-I} \simeq 1-2\frac{\mathcal{E}_o^2}{A_1^2}\,{}_2F_1(1,3/2;2;-2\mathcal{E}_o^2/A_1^2). \tag{3.32b}$$

From [Middleton, 1960, Eq. (A.1-40c)], the (gaussian) hypergeometric function in (3.32b) is explicitly

$${}_2F_1(1,3/2;2;-x^2) = (2/-x^2)(1+x^2)^{-1/2}\left\{1-\sqrt{1+x^2}\right\},$$

so that (3.32b) reduces explicitly to

$$\boxed{P_1(\mathcal{E}>\mathcal{E}_o)'_{B-I} \simeq \frac{1}{\sqrt{1+(2\mathcal{E}_o/A_1)^2}}, \quad (\alpha = 1)} \ , \tag{3.33}$$

which is a special case of the Hall model ($\theta_{Hall} = 2$), for envelopes [Spaulding and Middleton, 1975, Eq. (2.33)]. [Note that $P_1' \simeq \mathcal{E}_o^{-1}$, $\mathcal{E}_o \to \infty$, which checks with (3.31): in fact, both (3.31), (3.33) give $P_1' \simeq A_1/2\mathcal{E}_o$, as expected. Observe also that $P_1'(\mathcal{E}\geq\mathcal{E}_o=0)'_{B-I} = 1$, as required: P_1' is a proper P.D., although it is an inappropriate approximate form when $\Delta\sigma_G$ is at all comparable to $(b_{1\alpha}A_{1\alpha})^{1/\alpha}$, cf. (3.10a); it is also not applicable for very large $\mathcal{E}_o$, as explained earlier in A of Sec. 2.7 above. In any case, Hall-type pdf's and P.D.'s are not possible for Class A interference.]

Finally, we note that although the above PD's and pdf's, Eq. (3.26b), (3.31), exhibit the correct behaviour as $\mathcal{E}_0 \to 0$, , they are not scaled (in $\mathcal{E}_0$) properly, to provide the finite mean square needed, e.g. $\langle \mathcal{E}^2 \rangle = 1$, (cf. comments at the beginning of A above). Accordingly, as for P_{1-I} above generally, cf. (3.11b) et seq., we must replace $\mathcal{E}_0$ by $\mathcal{E}_0 N_I'$, where the scaling factor N_I' is determined, along with the four other parameters $(A_\alpha, \alpha, \Gamma_B', \Omega_{2B})$ of the distribution, by the procedure outlined in Section 6C following. [For the Hall model, $\alpha = 1$ here, and there are then only four parameter values $(N_I', A_\alpha, \Gamma_B', \Omega_{2B})$ to be established.]

4. PROBABILITY DENSITIES: $w_1(\mathcal{E})_{A,B}$

It is now a simple matter to determine the probability densities (pdf's) (pdf's) associated with the exceedance probabilities (PD's) derived in Section 3 preceeding. Because the PD's are continuous, at least through the second derivative ($0 \leq \mathcal{E} < \infty$), and because

$$w_1(\mathcal{E}) = -\left.\frac{dP_1}{d\mathcal{E}_0}\right|_{\mathcal{E}_0 \to \mathcal{E}} = \mathcal{E}\int_0^\infty \lambda J_0(\lambda\mathcal{E})\hat{F}_1(i a \lambda)\, d\lambda, \quad 0 \leq \mathcal{E} < \infty, \tag{4.1}$$

cf. (2.21)(3.1)(3.3)

we may apply this to (3.7), (3.11) and (3.17), etc., to obtain directly the desired pdf's. We have first:

4.1 Class A Interference:

From (3.7b) and (4.1) we find that

$$\boxed{w_1(\mathcal{E})_A \simeq e^{-A_A} \sum_{m=0}^{\infty} \frac{A_A^m}{m!} \frac{\mathcal{E} e^{-\mathcal{E}^2/2\hat{\sigma}_{mA}^2}}{\hat{\sigma}_{mA}^2}, \quad 0 \leq \mathcal{E}.} \tag{4.2}$$

Thus, as expected from our earlier result (3.7b), $w_1(\mathcal{E})_A$ (in its principal contribution*) is the weighted sum of rayleigh pdf's, whose variances $\hat{\sigma}_{mA}^2$ cf. (3.5), increase with order (m). Figs.(4.1)II and (4.2)II show $w_1(\mathcal{E})_A$ for various combinations of the controlling parameters A_A, Γ_A'. With A_A small the pdf's are seen to be highly nongaussian (e.g. nonrayleigh in $\mathcal{E}$), unless Γ_A' is very large, in which case the gaussian (e.g. rayleigh) component (here) dominates. As the Impulsive Index A_A gets larger, the pdf approaches the purely rayleigh form, cf. (2.57b). Also, for $\Gamma_A' > 0$, the pdf near $\mathcal{E} = 0$ has finite width, shouldering off into a broad, rather low level (in w_1) form as $\mathcal{E} \to \infty$, which represents the strongly non-rayleigh structure of this class of noise. The larger Γ_A', the wider and less "peaked" is the "spike" at $\mathcal{E} \gtrsim 0$, and the more "shoulder" there is to the rest of the pdf.

When $\Gamma_A' = 0$, e.g. $\sigma_G^2 = 0$, i.e., when there is no independent,

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* See the comments following Eq. (3.7b).

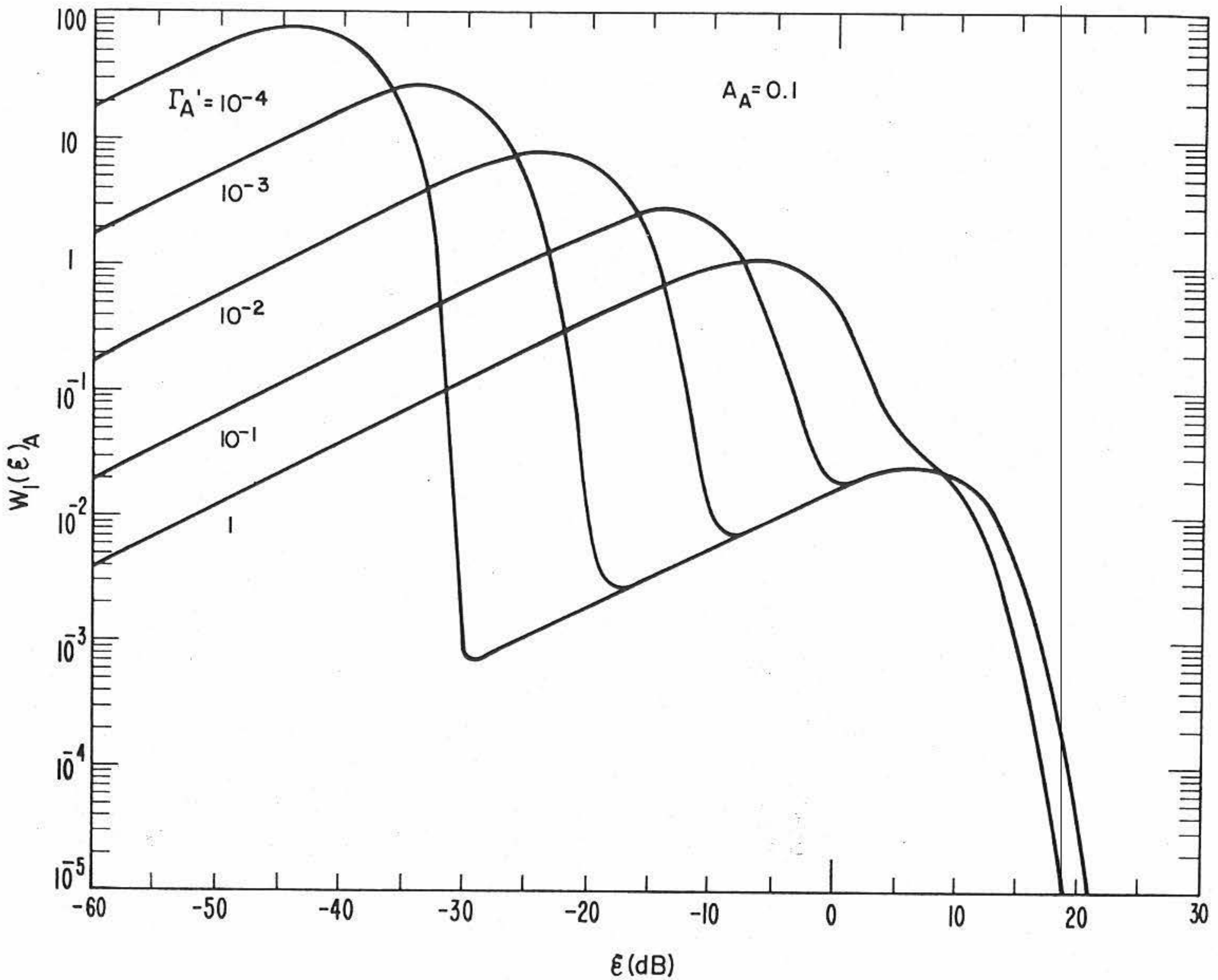

Figure 4.1 (II). The pdf, $w_1(\mathcal{E})_A$, of the envelope for Class A interference, calculated from eq. (4.2) for various Γ_A', given A_A [cf. fig. 3.1 (II) for $P_1(\mathcal{E} > \mathcal{E}_0)_A$].

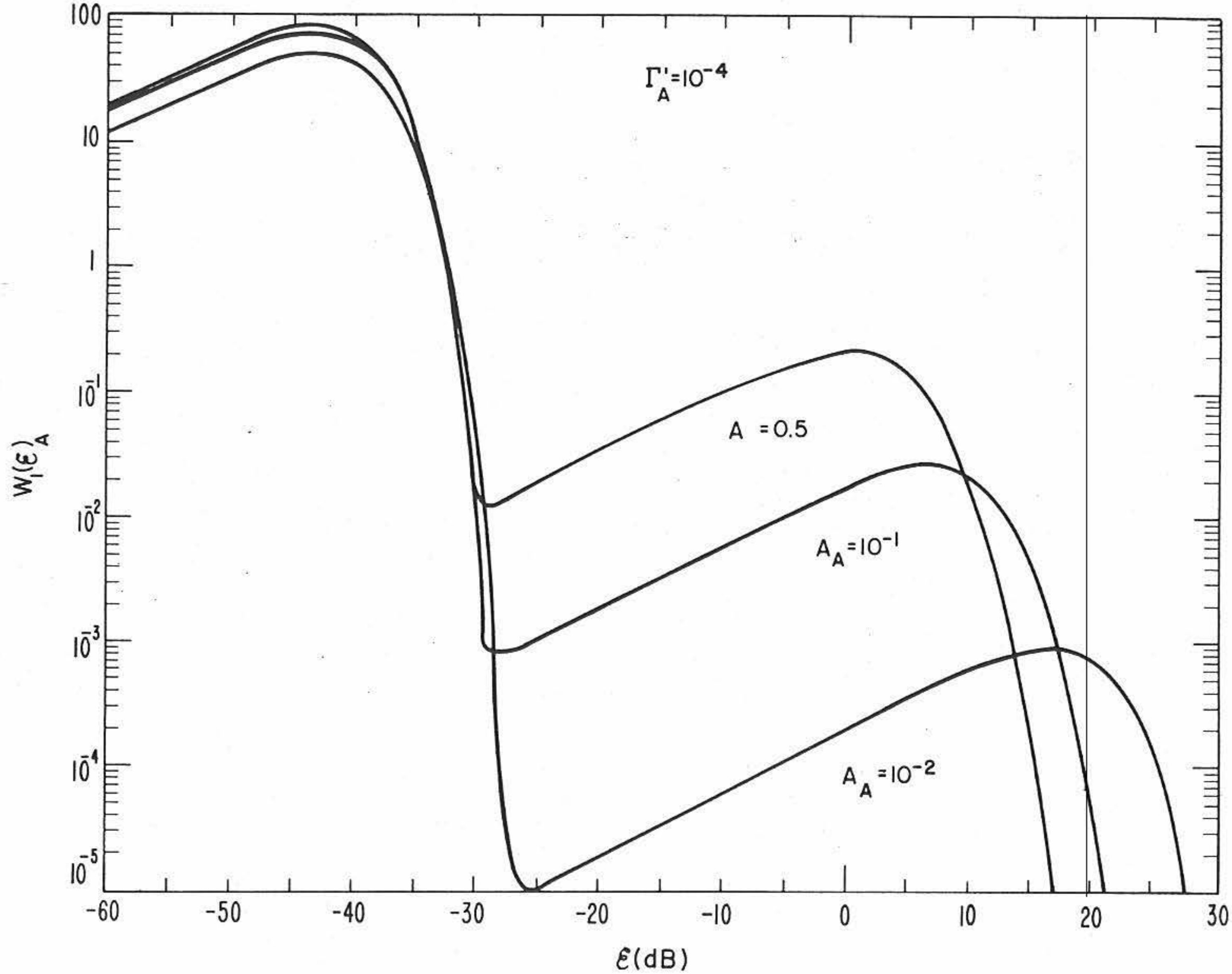

Figure 4.2 (II). The pdf, $w_1(\mathcal{E})_A$, of the envelope for Class A interference, calculated from eq. (4.2) for various A_A, given Γ_A' [cf. fig. 3.2 (II) for $P_1(\mathcal{E} > \mathcal{E}_0)_A$].

additive gaussian component to the interference, Eq. (4.2) reduces to

$$w_1(\mathcal{E})_A \simeq e^{-A_A}\left\{\delta(\mathcal{E}-0) + \sum_{m=1}^{\infty} \frac{A_A^{m+1}}{m!\ m}\ 2\mathcal{E}e^{-\mathcal{E}^2 A_A/m}\right\},\quad 0 \le \mathcal{E} < \infty), \qquad (4.2a)$$

and the "spike" at $\mathcal{E} = 0$ is truly a delta-function. The variance is now m/A_A. In this case we have an example of "holes in time": there is a nonzero probability that $\mathcal{E} = 0$, an idealized limiting case, since there is always in practice some system noise, which means an additive gaussian term, so that (4.2) applies, with $w_1(0)_A = 0$, of course.

4.2 Class B Interference:

As expected for our general canonical approximation [(3.11b) plus (3.17)], cf. (3.20) , we have also two relations for $w_1(\mathcal{E})_B$: $w_1(\mathcal{E})_{B-I}$ applies for small and intermediate values of $\mathcal{E}$, while $w_1(\mathcal{E})_{B-II}$ is appropriate for $\mathcal{E} \ge \mathcal{E}_B$. Again, $\mathcal{E}_B$ is a point of inflexion, or the "bend-over" point, where the Class A form, (3.17) applies, with, of course, the appropriate Class B parameters (A_B,Γ_B'), as determined analytically by the procedures described in (3.18), (3.19). Accordingly, we use (3.11b) in (4.1), to obtain specifically:

$$\boxed{\begin{aligned} &\hat{w}_1(\hat{\mathcal{E}})_{B-I} \equiv \\ &w_1(\mathcal{E})_{B-I} \simeq 2\hat{\mathcal{E}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{A}_\alpha^n\ \Gamma(1+\tfrac{n\alpha}{2})\,{}_1F_1(1+n\alpha/2;1;-\hat{\mathcal{E}}^2), \\ &\hat{\mathcal{E}} \equiv (\mathcal{E}N_I)/2G_B,\ \hat{A}_\alpha = A_\alpha/2^\alpha G_B^\alpha, \qquad (0 \le \mathcal{E} \le \mathcal{E}_B), \end{aligned}} \qquad (4.3)$$

and, formally, for large (but not too large) $\mathcal{E}$; from (3.15):

$$\hat{w}_1(\hat{\mathcal{E}})_{B-I} \simeq 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \hat{A}_\alpha^n \frac{\Gamma(1+\frac{n\alpha}{2})}{\Gamma(-\frac{n\alpha}{2})}\ \hat{\mathcal{E}}^{-n\alpha-1}\left\{1+\frac{(1+n\alpha/2)(1-n\alpha/2)}{1!\hat{\mathcal{E}}^2}+\dots\right\}$$

$$(0 \ll \mathcal{E} \le \mathcal{E}_B). \qquad (4.3a)$$

When $\mathcal{E} > \mathcal{E}_B$ we obtain at once from (3.17) in (4.1)

$$\boxed{w_1(\mathcal{E})_{B-II} \simeq \frac{e^{-A_B}}{4G_B^2} \sum_{m=0}^{\infty} \frac{A_B^m}{m!} \frac{\mathcal{E} e^{-\mathcal{E}^2/2\hat{\sigma}_{mB}^2}}{\hat{\sigma}_{mB}^2}, \quad (\mathcal{E}_B \leq \mathcal{E} < \infty),} \tag{4.4}$$

analogous to (4.2) in the Class A cases. Observe from (ii),(iv) of (3.18), and [(ii), (iv) of (3.19)], that $w_1(\mathcal{E})_B$, viz,:

$$\left.\begin{aligned} w_1(\mathcal{E})_B &= w_1(\mathcal{E})_{B-I} \quad , \qquad 0 \leq \mathcal{E} \leq \mathcal{E}_B \\ &= w_1(\mathcal{E})_{B-II} \quad , \qquad \mathcal{E}_B \leq \mathcal{E} \end{aligned}\right\} , \tag{4.5}$$

is continuous at $\mathcal{E} = \mathcal{E}_B$, with continuous first derivative, so that $w_1(\mathcal{E})_B$, as well as $P(\mathcal{E})_B$, has no break or "jump" at the bend-over point $\mathcal{E}_B$, where the two approximations are joined. Furthermore, unlike the Class A interference, when $\Gamma_B' = 0$ there are no "gaps in time", cf. (4.2a) vs. (4.3): there is always a non-zero probability (density) for $\mathcal{E} = 0$. Figs.(4.3)II,(4.3)II show typical curves of $w_1(\mathcal{E})_B$, analogous to Figs. (3.6)II, (3.7)II for the P.D.

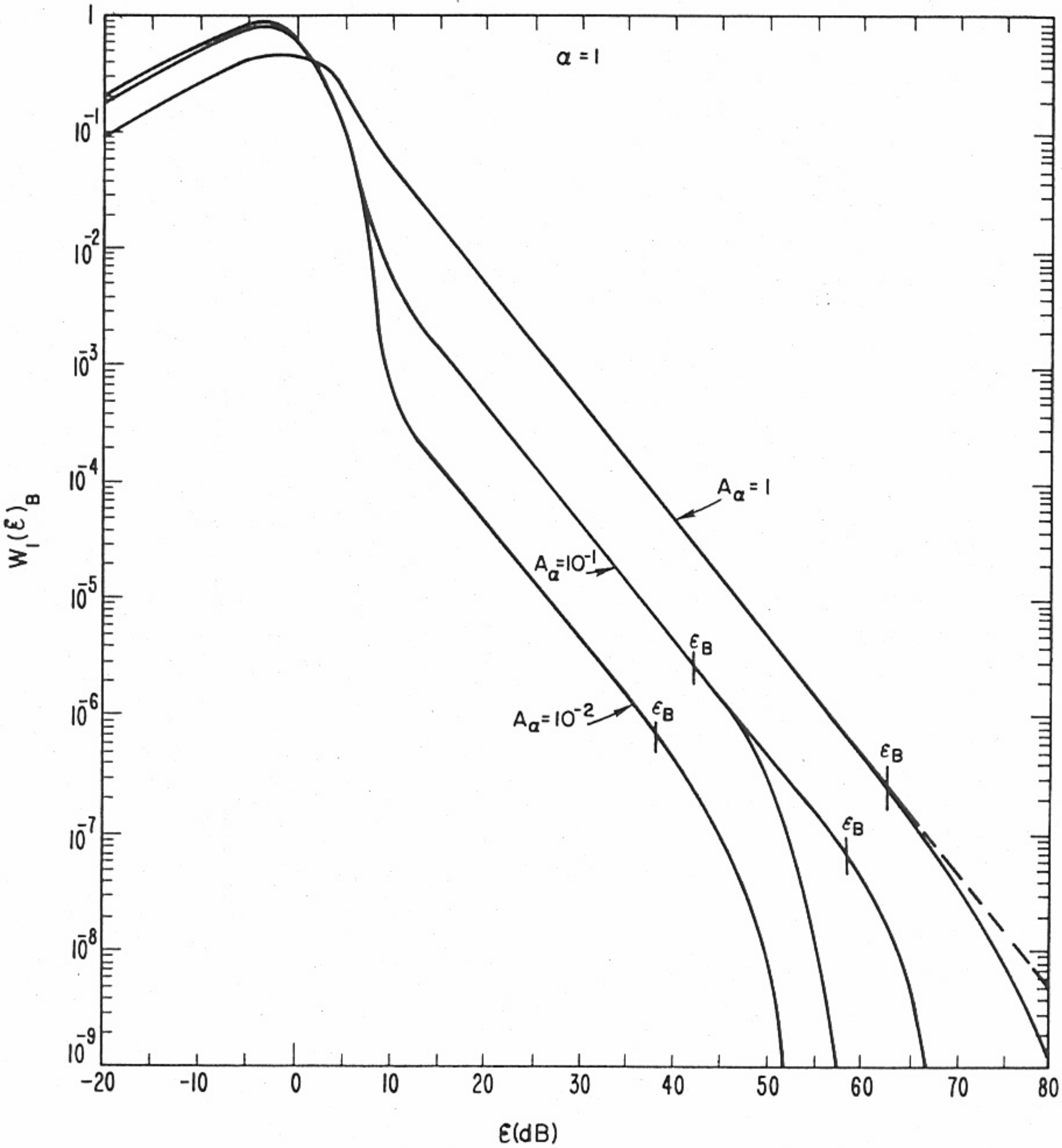

Figure 4.3 (II). The (complete) pdf $w_1(\mathcal{E})_B$, eq. (4.5), of the envelope for Class B interference, calculated from eqs. (4.3, 4.4) for various A_α, given α [cf. (3.19)]. [See fig. 3.6 (II) for the associated $P_1(\mathcal{E} > \mathcal{E}_0)_B$ and parameter values.]

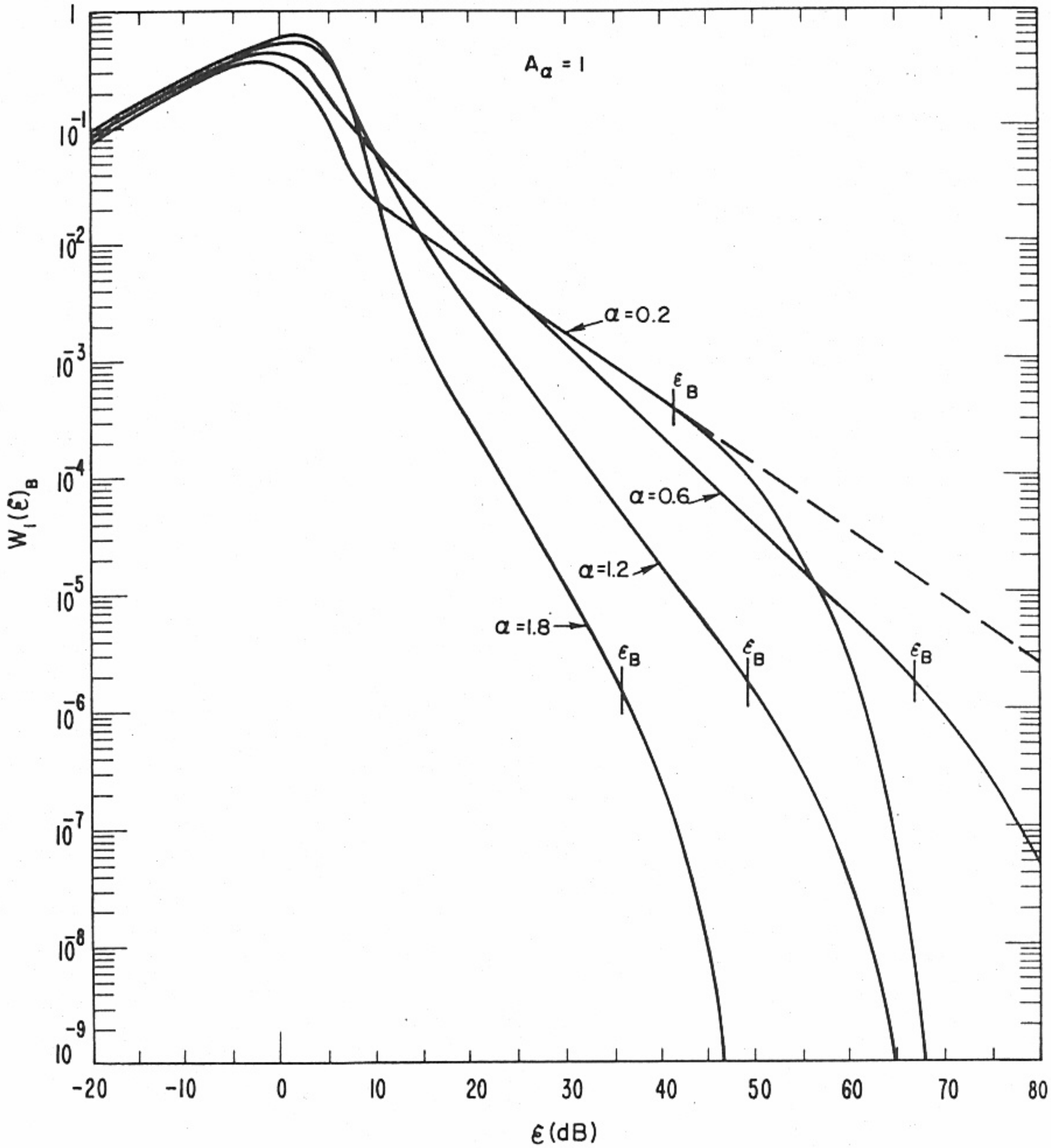

Figure 4.4 (II). The (complete) pdf, $w_1(\mathcal{E})_B$, eq. (4.5), of the envelope Class B interference, calculated from eqs. (4.3, 4.4) for various α, given A_α [cf. (3.19)]. [See fig. 3.7 (II) for the associated $P_1(\mathcal{E} > \mathcal{E}_0)_B$ and parameter values.]

5. MOMENTS

The general first-order moments $\langle \mathcal{E}^\beta \rangle$, $(0 \leq \beta < \infty)$, are now easily obtained from the results of the preceeding Section. Since

$$\left\langle \mathcal{E}^\beta \right\rangle \equiv \int_0^\infty w_1(\mathcal{E})\mathcal{E}^\beta d\mathcal{E}, \qquad 0 \leq \beta < \infty \quad , \tag{5.1}$$

(β real and nonnegative), we may apply (4.2) for Class A interference, and (4.5), with (4.3), (4.4), for Class B noise, respectively.

5.1 Existence and Direct Calculation (Approximate Forms):

For Class A interference we get directly

$$\left\langle \mathcal{E}^\beta \right\rangle_A \cong e^{-A_A} \sum_{m=0}^{\infty} (2\hat{\sigma}_{mA}^2)^{\beta/2} \Gamma(\tfrac{\beta}{2}+1) A_A^m / m!$$

$$= e^{-A_A} \Gamma(\tfrac{\beta}{2}+1) \sum_{m=0}^{\infty} \left(\frac{m/A_A + \Gamma_A'}{1+\Gamma'} \right)^{\beta/2} \frac{A_A^m}{m!} \quad , \tag{5.2}$$

cf. (3.5). The sum in (5.2) is clearly finite, since by Stirling's theorem ($m! \simeq m^m e^{-m}\sqrt{2\pi m}$) for sufficiently large m ($\gg A_A \Gamma_A'$) the summand is dominated by $A_A^m / m^{m+1-\beta}$. Accordingly, all (finite) moments* exist for Class A interference, and are given by (5.2), approximately. [We recall that we consider only the principal development of P_{1A}, w_{1A}, cf. (3.7), (4.2) above.] Typical moments here are, from (5.2):

$$\left\langle \mathcal{E}^0 \right\rangle_A = 1 \quad \text{, as required;}$$

---

* Of the envelope, and of the instantaneous amplitude [cf. Middleton, 1974, Section 4.5], by the same argument.

$$\left\langle \mathcal{E} \right\rangle_A = e^{-A_A} \frac{\sqrt{\pi}}{2} \sum_{m=0}^{\infty} \left(\frac{m/A_A + \Gamma'_A}{1+\Gamma'_A} \right)^{1/2} \frac{A_A^m}{m!} \quad ;$$

$$\left\langle \mathcal{E}^2 \right\rangle_A \cong e^{-A_A} \sum_{m=0}^{\infty} \left(\frac{m/A_A + \Gamma'_A}{1+\Gamma'_A} \right) \frac{A_A^m}{m!} \quad = 1, \text{ (as required)}. \tag{5.3}$$

For the Class B interference we have, from (4.3), (4.4) in (5.1):

$$\left\langle \mathcal{E}^{\beta} \right\rangle_B = \int_0^{\mathcal{E}_B} \mathcal{E}^{\beta} w_1(\mathcal{E})_{B-I} d\mathcal{E} + \int_{\mathcal{E}_B}^{\infty} \mathcal{E}^{\beta} w_1(\mathcal{E})_{B-II} d\mathcal{E} \tag{5.4a}$$

$$\cong 2 \sum_{n=o}^{\infty} \frac{(-1)^n}{n!} \hat{A}_{\alpha}^n \Gamma(1+n\alpha/2) \int_0^{\hat{\mathcal{E}}_B} \mathcal{E}^{\beta} \; {}_1F_1(1+n\alpha/2;1;-\hat{\mathcal{E}}^2) d\mathcal{E}$$

$$+ \frac{e^{-A_B}}{4G_B^2} \sum_{m=0}^{\infty} \frac{A_B^m}{m!} \int_{\mathcal{E}_B}^{\infty} \mathcal{E}^{\beta+1} e^{-\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2} d\mathcal{E}/\hat{\sigma}_{mB}^2 \, . \tag{5.4b}$$

[The integration of the hypergeometric function directly from its series form yields

$$\frac{N_I}{2G_B} \int_0^{\mathcal{E}_B} \mathcal{E}^{\beta+1} \, {}_1F_1(1+n\alpha/2;1;-\hat{\mathcal{E}}^2) d\mathcal{E} = \sum_{k=0}^{\infty} \frac{(-1)^k \mathcal{E}_B^{\beta+2k+2} (1+\alpha n/2)_k}{(k!)^2 (\beta+2k-2)(2G_B)^{2k}} \left(\frac{N_I^{2k+1}}{2G_B} \right) \tag{5.5a}$$

which is probably the most direct and convenient form for numerical integration here.] In addition, the second integral in (5.4b) may be expressed as an incomplete gamma function, I_c, which is tabulated [K. Pearson, 1951], e.g.

$$\int_{\mathcal{E}_B}^{\infty} \mathcal{E}^{\beta+1} e^{-\mathcal{E}^2/2\hat{\sigma}_{mB}^2} d\mathcal{E}/\hat{\sigma}_{mB}^2 = (2\hat{\sigma}_{mB}^2)^{\beta/2} \left\{ 1 - I_C[\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2; 1+\beta/2] \right\} , \tag{5.5b}$$

where I_C is defined by

$$I_C(x;\gamma) \equiv \frac{1}{\Gamma(\gamma)} \int_0^x y^{\gamma-1} e^{-y} dy \; . \tag{5.5c}$$

Again, direct integration of the integral itself in (5.5b) is probably the most convenient numerical procedure.] Clearly, by the same argument used above for the Class A noise [cf. (5.2) et seq.], all (finite) moments exist for Class B interference, as well. Some typical moments here are, from (5.4):

$$\left\langle \mathcal{E}^o \right\rangle_B = 1, \text{ as required;} \tag{5.6a}$$

$$\left\langle \mathcal{E} \right\rangle_B \simeq 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{A}_\alpha^n \, \Gamma(1+n\alpha/2) \int_0^{\mathcal{E}_B} \hat{\mathcal{E}}\, \mathcal{E} \; {}_1F_1(1+n\alpha/2;1;-\hat{\mathcal{E}}^2) d\mathcal{E}$$
$$+ \frac{e^{-A_B}}{4G_B^2} \sum_{m=0}^{\infty} (2\hat{\sigma}_{mB}^2)^{\frac{\beta+1}{2}} \frac{A_B^m}{m!} \left\{ 1 - I_C[\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2 ; 3/2+\beta/2] \right\} \; ; \tag{5.6b}$$

$$\left\langle \mathcal{E}^2 \right\rangle_B \simeq 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{A}_\alpha^n \Gamma(1+n\alpha/2) \int_0^{\mathcal{E}_B} \hat{\mathcal{E}}\, \mathcal{E}^2 \; {}_1F_1(1+n\alpha/2;1;-\hat{\mathcal{E}}^2) d\mathcal{E}$$
$$+ \frac{e^{-A_B}}{4G_B^2} \sum_{m=0}^{\infty} (2\hat{\sigma}_{mB}^2)^{\beta/2+1} \frac{A_B^m}{m!} \left\{ 1 - I_C[\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2 ; 2+\beta/2] \right\} < \infty \; ; \tag{5.6c}$$

[For numerical calculation we may also evaluate the integrals (left member of (5.5b)) directly, replacing β by $\beta+1$, $\beta+2$, respectively for the mean and mean square, (5.6b), (5.6c).]

As required from physical considerations, e.g. finite energy in the Class B (and A) interference, the second moment, in particular, is itself finite. This is not the case if we use the approximate forms $w_1(\mathcal{E})_{B-I}$, or $w_1'(\mathcal{E})_{B-I}$ $(= -dP_{1(B-I)}'/d\mathcal{E}$, cf. (3.21) et seq.), for all values of the envelope. Then, we have, in effect

$$\left\langle \mathcal{E}^\beta \right\rangle \lesssim \int_0^{\mathcal{E}_B} \mathcal{E}^\beta \hat{w}_1(\hat{\mathcal{E}})_{B-I} d\mathcal{E} + \frac{2\Gamma(1+\alpha/2)}{\Gamma(-\alpha/2)} A_\alpha \int_{\mathcal{E}_B}^{\infty} \mathcal{E}^{\beta-\alpha-1} d\mathcal{E} \; , \tag{5.7}$$

this last term from (3.31). The second term of (5.7) is finite only if $0 \leq \beta < \alpha$ (<2). Accordingly, no β moments exist for the Class (B-I) distributions, unless β is less than the spatial density - propagation parameter α. Thus, $\langle \mathcal{E}^2 \rangle_{B-I} \to \infty$, in contradiction to the physical situations we are attempting to model. In some cases (e.g. atmospheric noise) w_{B-I}, P_{B-I} are quite satisfactory for even very large values of $\mathcal{E}$, cf. Figs. (3.3) II, (3.4) II, but there is always some finite $\mathcal{E}_B$ beyond which a suitable form of w_{B-II}, or P_{B-II} must be used, in keeping with the bounded nature of all the moments.

5.2 Class A and B Moments: Exact Forms (Even Moments Only):

As we have seen above in Sec. (5.1) all (first-order) Class A and B moments of the envelope (and hence of the instantaneous amplitude) exist and are given, an approximate form, by Eqs. (5.2), (5.4). These results are approximate, albeit good ones.

However, an alternative development is possible, which can provide us with exact expressions, for the even-order moments.* For this we use (2.24b) and take its $[(-i)^k \, d^k/dr^k]$ derivative at r=0, to get

$$\left[(i)^k \frac{d^k}{dr^k} \hat{F}_1(ir)\right]_{r=0} = \left\langle E^k \cos^k \psi \right\rangle_{E,\psi} = \left\langle E^k \right\rangle \left\langle \cos^k \psi \right\rangle \text{ , } k = 0,1,2,\ldots, \tag{5.8}$$

this last, since $W_1(E,\psi) = W_1(E)W_1(\psi)$, cf. (2.20), (2.21), a result of the narrow-band nature of the output from the (aperture x RF x IF) stages of the receiver. Since $W_1(\psi)$ is uniform in $(0,2\pi)$, all k-odd moments of the phase vanish, e.g. $\langle \cos^k \psi \rangle = 0$; k = 1,3,5,..., and only the k-even moments remain. With the help of

$$\left\langle \cos^{2k} \psi \right\rangle_\psi = {}_{2k}C_k / 2^{2k} = (2k)!/2^{2k}(k!)^2 \tag{5.9}$$

[Middleton, 1960, Eq. (5.26)] we accordingly obtain from (5.8) the general relation

---

* No such results are available for the odd-order cases; one must use the approximate forms (5.2), (5.4).

$$\boxed{\left\langle E^{2k}\right\rangle = \frac{(k!)^2 2^{2k}}{(2k)!}\left[(-1)^k \frac{d^{2k}}{dr^{2k}}\hat{F}_1(ir)\right]_{r=0}} \tag{5.10}$$

for the even moments, <u>when they exist</u> (as they do here, but see the comments below in Section 5.3). In normalized form [with the aid of (3.3)] we can write (5.10) equivalently as

$$\boxed{\left\langle \mathcal{E}^{2k}\right\rangle = \frac{(k!)^2 2^{2k}}{(2k)!}\left[(-1)^k \frac{d^{2k}}{d\lambda^{2k}}\hat{F}_1(ia\lambda)\right]_{\lambda=0}} \; . \tag{5.10a}$$

Our general result (5.10), (5.10a) is equivalent to the procedures used by Furutsu and Ishida [1960, Section 6] and Giordano [1970, Appendix II, p. 175 et seq.], which is derived in our study by a different process.

<u>A. Exact Class A Even Moments:</u>

For Class A, even-order moments, including an independent gaussian component, the <u>exact</u> form is now obtained by using (2.50) in (5.10) or (5.10a), rather than from the approximate relations (2.77), (2.78). The simplest procedure here is to expand the c.f. $\hat{F}_1(ia\lambda)_A$ as a power series in λ^2, which is permitted, since the integral (in the exponent) is a definite integral, continuous in λ^2. The functional <u>form</u> of $\hat{F}_1(ia\lambda)_A$ is seen to be precisely that of the c.f. $F_1(i\xi)_{A=P+G}$ derived in the earlier study for the statistics of the instantaneous amplitudes (Class A noise), [Middleton, 1974, Section 4]. Accordingly, we use that expansion of F_1 to write at once here (exactly)

$$\hat{F}_1(ia\lambda)_A = 1 - \frac{a^2\lambda^2}{2!}\Omega_{2A}(1+\Gamma_A') + \frac{a^4\lambda^4}{4!}\left[\frac{3}{2}\Omega_{4A} + 3\Omega_{2A}^2(1+\Gamma_A')^2\right]$$

$$- \frac{a^6\lambda^6}{6!}\left[\frac{5}{2}\Omega_{6A} + \frac{45}{2}\Omega_{4A}\Omega_{2A}(1+\Gamma_A') + 15\Omega_{2A}^3(1+\Gamma_A')^3\right] + 0(a^8\lambda^8), \tag{5.11a}$$

$$= 1 - \frac{1}{2!}\frac{\lambda^2}{2} + \frac{\lambda^4}{4!}\left[\frac{3\Omega_{4A}}{8\Omega_{2A}^2(1+\Gamma_A')^2} + \frac{3}{4}\right] - \frac{\lambda^6}{6!}\left[\frac{5\Omega_{6A}}{16\Omega_2^3(1+\Gamma_A')^3} + \frac{45\Omega_{4A}}{16\Omega_{2k}^2(1+\Gamma_A')^2} + \frac{15}{8}\right]$$

$$+ \ldots, \tag{5.11b}$$

where we have

$$\Omega_{2k\text{-}A} \equiv A_A\left\langle \hat{B}_{oA}^{2k}\right\rangle / 2^k, \text{ (cf. (2.75d)) ; } \Gamma_A' \equiv \sigma_G^2/\Omega_{2A} \text{ , cf. (3.1a),}$$

with

$$\langle\,\rangle \equiv \left\langle \int_0^{z_o} (\,)dz \right\rangle_{[\theta=z_o, A_o, e_{o\gamma}, \mathcal{A}_{RT}]} \text{ , cf. (2.64c), (2.65), (2.75d) for } \left\langle \hat{B}_{oA}^{2k}\right\rangle .$$

Applying (5.11) to (5.10a) and observing that the expression in the square brackets [ ] in (5.10a) is precisely the coefficient of $(-1)^k(a\lambda)^{2k}/(2k)!$ in (5.11a) [or of $(-1)^k\lambda^{2k}/(2k)!$ in (5.11b)], we obtain

$$[\langle \mathcal{E}^0\rangle_A = 1] \text{ (as expected) ;} \tag{5.12a}$$

$$\langle \mathcal{E}^2\rangle_A = 2a^2\Omega_{2A}(1+\Gamma_A') = 1 \text{ (as expected) ;} \tag{5.12b}$$

$$\langle \mathcal{E}^4\rangle_A = \frac{8}{3}\, a^4[\tfrac{3}{2}\,\Omega_{4A}+3\Omega_{2A}^2(1+\Gamma_A')^2] = \frac{\Omega_{4A}}{\Omega_{2A}^2(1+\Gamma_A')^2} + 2 \text{ ;} \tag{5.12c}$$

$$\langle \mathcal{E}^6\rangle_A = \frac{16a^6}{5}\,[\tfrac{5}{2}\,\Omega_{6A} + \tfrac{45}{2}\,\Omega_{4A}\Omega_{2A}(1+\Gamma_A') + 15\Omega_{2A}^3(1+\Gamma_A')^3]$$

$$= \frac{\Omega_{6A}}{\Omega_{2A}^3(1+\Gamma_A')^3} + \frac{9\Omega_{4A}}{\Omega_{2A}^2(1+\Gamma_A')^2} + 6 \text{ ; etc.} \tag{5.12d}$$

(This is given in unnormalized form, e.g. with $\mathcal{E}/a = E$, by (5.12) on replacing $\mathcal{E}$ by E and deleting a therein.) These results (5.12) are to be compared with the approximate (and sometimes exact) forms (5.3). For example, observe that when $\Gamma_A' \to 0, \infty$, $\langle \mathcal{E}^2\rangle = 1$, as required, cf. (5.12b), with similar equivalences for $\langle \mathcal{E}^4\rangle$, etc., from (5.2) vis-à-vis (5.12c), (5.12d), etc. For intermediate values of A_A, Γ_A', we may expect modest departures from the exact values above. Finally, using (5.11c) in (5.12) we can write alternatively

$$\langle \mathcal{E}^0 \rangle_A = 1 \; ; \quad \langle \mathcal{E}^2 \rangle_A = 1 \; ;$$

$$\langle \mathcal{E}^4 \rangle_A = 2 + \langle \hat{B}^4_{oA} \rangle / A_A \langle \hat{B}^2_{oA} \rangle^2 (1+\Gamma'_A)^2 ;$$

$$\langle \mathcal{E}^6 \rangle_A = 6 + \frac{9\langle \hat{B}^4_{oA} \rangle}{A_A \langle \hat{B}^2_{oA} \rangle^2 (1+\Gamma'_A)^2} + \frac{\langle \hat{B}^6_{oA} \rangle}{A_A^2 \langle \hat{B}^2_{oA} \rangle^3 (1+\Gamma'_A)^3} \; ; \text{ etc.} \tag{5.13}$$

which shows how the (normalized moments behave as the Impulsive Index $A_A \to \infty$, or as the independent gaussian component becomes dominant ($\Gamma'_A \to \infty$).

For the odd moments of $\mathcal{E}$(or E) our procedure above, of course, is not applicable, and we must go directly to the calculation on the pdf, $w_1(\mathcal{E})_A$, cf. (5.1), (5.2), for k=1,3,5,... .

B. Exact Class B Even Moments:

The relations (5.10), (5.10a) apply here also, but the explicit differentiation of $F_1(ia\lambda)_B$, based on (2.51) cannot make direct use of a power series expansion of the integrand in the exponential, because the integral is now an improper integral $(0,\infty)$ which is not uniformly convergent (in λ) over the entire domain of integration [Courant, 1936,II Sec. 4, Chapter 4, p. 307 et seq.], so that term-wise expansion (in λ) of the integrand, as for Class A interference above, is not permitted. However, let us temporarily consider the case where $(z_o)_{max} < \infty$ (e.g., output signals of finite duration). Then, the term-wise expansion is permitted, as the integral is now both proper and uniformly convergent; (in fact, the resulting c.f. belongs formally to Class A). We proceed as in Class A above and next apply $\lim (z_o)_{max} \to \infty$ to the c.f., e.g. $F_1(ia\lambda)_B = \lim_{(z_o)_{max} \to \infty} F_1(ia\lambda | (z_o)_{max} < \infty)_{"B"}$, and hence to each term of (5.11), (5.12), etc., now specialized to the Class B parameters, A_B, Γ'_B, $\hat{B}_{oB}$, etc. Thus, (5.11) applies again here, with $\Gamma'_A \to \Gamma'_B$, $A_A \to A_B$, $\hat{B}_{oA} \to \hat{B}_{oB}$, etc. We obtain the analogue of (5.12), for example:

$$
\left.\begin{aligned}
&\left.\begin{aligned}\langle \mathcal{E}^0\rangle_B &= 1;\\ \langle \mathcal{E}^2\rangle_B &= 1,\end{aligned}\right\}\text{ as expected;}\\
&\langle \mathcal{E}^4\rangle_B = \Omega_{4B}/\Omega_{2B}^2(1+\Gamma_B')^2 + 2;\\
&\langle \mathcal{E}^6\rangle_B = \Omega_{6B}/\Omega_{2B}^3(1+\Gamma_B')^3+9\Omega_{4B}/\Omega_{2B}^2(1+\Gamma_B')^2+6;\text{ etc.}
\end{aligned}\right\}, \tag{5.14}
$$

where specifically,

$$
\Omega_{2k-B} \equiv \frac{A_B\left\langle \hat{B}_{oB}^{2k}\right\rangle}{2^k}\,[\underset{\sim}{\theta}_B' = A_o,\, e_{o\gamma},\, \mathcal{A}_{RT}]_B \quad ; \quad \langle\ \rangle_{\underset{\sim}{\theta}_B'} \equiv \left\langle \int_0^\infty (\ )dz\right\rangle_{\underset{\sim}{\theta}_B'}\ , \tag{5.14a}
$$

cf. (2.87d) and $\hat{B}_{oB}$ in (2.87c). Again, for the odd-moments (k=1,3,5,...) we must use the approximate forms (5.4), (5.6), etc.

5.3 Remarks:

For Class B (and $\therefore$ Class C) interference ($0 < \alpha < 2$), when (2.89) is used as an approximation for the c.f., it is clear that if we use (2.89) in (5.10a), then $\langle \mathcal{E}^2\rangle \to O(\lambda^{\alpha-2})_{\lambda=0}\to \infty$: the second moment does not exist. Of course, this divergence is simply the consequence of the inadequate approximation, a behaviour which is alleviated by the alternative approach using the results of Section 4, Eqs. (4.3)-(4.5) in (5.1), cf. (5.4)-(5.6).

Of perhaps greater interest is to note that, in terms of our general classification [Sec. (2.3) and Sec.(2.5-3)], Giordano and Haber [1970,1972], in effect, postulate a finite period of observation (0,T) for each member of the ensemble, e.g. Eq. (2.36) above is in force. This is equivalent to Class A operation, since it amounts to an abrupt truncation of the basic signal waveform $u_0(z)$ as emitted from the ARI stages of the typical narrow-band receiver, cf. Fig.(2.1)II. This, in turn, means that the receiver bandwidth is large enough vis-à-vis that of the input to pass it with negligible transients, a defining characteristic of Class A noise. Then all moments exist [cf. Sec. (5.1)] and the proper approximation for the

PD is (3.7b). This Class A, or truncated case, goes over into a Class B model as the observation period (0,T) becomes large vis-à-vis receiver bandwidth, e.g., as $T\Delta f_{ARI} >> 1$. The type of approximate c.f. employed is then that given by (2.89), or more suitably, (2.90), which includes an independent gaussian component. These, as we have seen [cf. Sec. (3.2)] yield satisfactory approximations for small and intermediate ranges of envelope $\mathcal{E}$, or thresholds $\mathcal{E}_0$, but fail at some point ("large" $\mathcal{E}$, $\mathcal{E}_0$) to give the more rapid convergence needed to insure the physically required finite moments of all (positive) orders, cf. Sec. (5.1). Thus, the results of Giordano and Haber [1970,1972] (for suitably large $T\Delta f_{ARI}$), while practically useful as long as the statistics of very large values of the envelope are not demanded, are analytically incomplete as Class B models of the full range of possible values of the random envelope $\mathcal{E}$ and exceedance probability $P_1(\mathcal{E} > \mathcal{E}_0)_B$.

On the other hand, the important analysis of Furutsu and Ishida [1960], which represents a subclass of our Class B model in that a specific emitted waveform $[u_0(z)_B]$ is chosen, i.e. an exponential $\sim e^{-\alpha z}$, and several, particular spatial distributions of sources with a given propagation law $(\sim 1/\lambda)$ are assumed, along with an exponential distribution of input signal amplitudes, does yield analytic forms of the PD and pdf which permit the existence of all orders of envelope moments, and which conform closely to the statistics of the atmospheric data studied therein. The approach of Furutsu and Ishida [1960] is similar to ours in that approximate c.f.'s are obtained, suitable for the small and large values of $\mathcal{E}$(and $\mathcal{E}_0$), while the intermediate ranges (of $\mathcal{E}$) are evaluated by numerical techniques. The canonical methods of our approach, however, are not invoked. Of course, neither Furutsu and Ishida, nor Giordano, and others, consider or distinguish Class A interference, which is a new category, as far as its statistical-physical description is concerned, considered originally by the author [Middleton, 1973, 1974].

6. DETERMINATION OF THE BASIC FIRST-ORDER PARAMETERS

In this section we outline procedures for determining the basic parameters of Class A and Class B interference models developed in the preceeding sections. A variety of overlapping procedures is available. We shall select what appears at this stage of the study to be the most direct and/or convenient, (later efforts may suggest modifications, for particular situations).

We begin with:

A. Class A Interference:

The first-order PD (and pdf) are governed by three parameters. It is convenient to distinguish two levels of parametric description: the first level, which we shall call "Basic-I", consists of global parameters, which appear directly in the expression for the P.D., cf. Eq. (3.7), and the second, or "Basic-II" level, contains the associated generic parameters, which are defined directly in terms of the underlying statistical-physical model. The two groups, as we shall see, overlap to some extent. Table 6.1 below gives the global and generic parameters of Class A interference:

Table (6.1): Class A Parameters

$$\left.\begin{array}{lll} \underline{\text{Basic I:}} & \underline{\text{Global:}} & (A_A, \Gamma'_A, \Omega_{2A}) \rightarrow \{\text{Practical Global: } (A_A, \Gamma'_A, K_A)\} \\ \underline{\text{Basic II:}} & \underline{\text{Generic:}} & (A_A, \sigma_G^2, \langle \hat{B}_{oA}^2 \rangle) \end{array}\right\} \tag{6.1}$$

The generic and global parameters are related by

$$\sigma_G^2 = \Omega_{2A}\Gamma'_A \; ; \; \langle \hat{B}_{oA}^2 \rangle = 2\Omega_{2A}/A_A \quad , \quad [\text{Eq. (3.1a)}] \ . \tag{6.2}$$

Furthermore, the intensity of the independent gaussian component is

$$\sigma_G^2 = \sigma_R^2 + \sigma_E^2 \ , \ [\text{Eqs. (2.47)}], \tag{6.3}$$

where σ_R^2 is the intensity of receiver noise (at the output of the initial ARI-stages of the receiver) and σ_E^2 is the intensity of the independent external gaussian component, if any, likewise observed at the output of the ARI-stages of the receiver. By blocking the input to the receiver, (i.e., insuring that $\sigma_E^2 = 0$) one obtains σ_R^2, at the ARI output. Consequently, as σ_E^2 is found by actual reception in the (here) Class A noise environment, one then at once determines σ_E^2 from (6.3).

Because we do not a priori know the normalizations (3.1) by which the threshold E_0 and the envelope E are scaled, it is necessary to convert our analytic expressions (3.7) for the PD, for example, into forms more directly conformable to experimental evaluation. For this purpose we write*

$$\mathcal{E}' \equiv E/\sqrt{2\hat{\sigma}_G^2} = \mathcal{E}\sqrt{\Omega_{2A}(1+\Gamma_A')/\hat{\sigma}_G^2} = \mathcal{E}/K_A; \quad \mathcal{E}_0' = \mathcal{E}_0/K_A, \tag{6.4a}$$

with

$$K_A \equiv [\hat{\sigma}_G^2/\Omega_{2A}(1+\Gamma_A')]^{1/2} = [\hat{\sigma}_G^2/(\Omega_{2A}+\sigma_G^2)]^{1/2}, \tag{6.4b}$$

the new conversion factor, between $\mathcal{E}'$ and $\mathcal{E}$, where $\hat{\sigma}_G^2$ is an a priori determined reference quantity, used to scale the (absolute) values E_0, E, etc.

To obtain the generic parameters (A_A, σ_G^2, $\hat{B}_{0A}^2$) we first must determine the global parameters ($A_A, \Gamma_A', \Omega_{2A}$). Practically, this means we must initially find the "practical" global quantities (A_A, Γ_A', K_A), cf. Table (6.1) above, and then use (6.4b) to eliminate the conversion factor K_A. Three relations involving the practical global parameters are needed. Perhaps the simplest are the first and second moments of $\mathcal{E}'$, and the PD of $\mathcal{E}'$ in the rayleigh region ($\mathcal{E}_0' \ll 1$), where the slope ($dP_{1A}/d\mathcal{E}_0'$) is constant, cf. Figs. (3.1,3.2)II. Accordingly, from the exact expression (5.12b) and (6.4a) we write

$$\langle(\mathcal{E}')^2\rangle_A = \langle\mathcal{E}^2\rangle_A/K_A^2 = 1/K_A^2 = \Omega_{2A}(1+\Gamma_A')/\hat{\sigma}_G^2, \tag{6.5}$$

which gives us K_A and hence $\Omega_{2A}(1+\Gamma_A')$ in terms of the known $\hat{\sigma}_G^2$ and $\langle(\mathcal{E}')^2\rangle_A$,

*Of course, one can always measure $\langle E^2\rangle_A = 2\Omega_{2A}(1+\Gamma_A')$ and then normalize, so that $K_A=1$.

this last by measurement in practice. From the approximate expression for $\langle \mathcal{E}' \rangle_A$, viz. (5.3) with (6.4a), we obtain

$$\langle \mathcal{E}' \rangle_A = \langle \mathcal{E} \rangle_A / K_A \cong K_A^{-1} \left[e^{-A_A} \frac{\sqrt{\pi}}{2} \sum_{m=0}^{\infty} \frac{(m/A_A + \Gamma_A')^{1/2}}{(1+\Gamma_A')^{1/2}} \frac{A_A^m}{m!} \right] , \tag{6.6}$$

and from (3.7b) specialized to the rayleigh region, e.g. (3.8), we have

$$P_1(\mathcal{E}' > \mathcal{E}_o')_A \cong 1 - \left[e^{-A_A} \sum_{m=0}^{\infty} \frac{(1+\Gamma_A') A_A^m}{(\Gamma_A' + m/A_A) m!} \right] (\mathcal{E}_o' K_A)^2 \quad . \tag{6.7}$$

In practice, of course, $\langle \mathcal{E}' \rangle_A$, $\langle (\mathcal{E}')^2 \rangle_A$, and $P_1(\mathcal{E}' > \mathcal{E}_o')$ are estimated from the experimentally derived data, i.e. $\langle \mathcal{E}' \rangle_A$, $\langle (\mathcal{E}')^2 \rangle_A$, and $P_1(\mathcal{E}' > \mathcal{E}_o')_A$ are respectively replaced by their estimates from the necessarily finite empirical data, so that (6.5)-(6.7) are three relations for joint estimation of the "practical" global parameters (A_A, Γ_A', K_A), a proceedure requiring a modest amount of computational assistance, particularly when the expressions in brackets [ ] have been programmed. With the help of (6.4b) for K_A involving Ω_{2A}, Γ_A' we next get directly the (estimates of the) global quantities $(A_A, \Gamma_A', \Omega_{2A})$. Then it is a simple matter to use (6.2) to obtain finally the (estimates of the) desired generic parameters $(A_A, \sigma_G^2, \langle \hat{B}_{oA}^2 \rangle)$. The desired estimates to be used in (6.5)-(6.7) are

$$\langle \mathcal{E}' \rangle_A \to \frac{1}{n} \sum_{i=1}^{n} \mathcal{E}_i' \; ; \quad \langle (\mathcal{E}')^2 \rangle_A \to \frac{1}{n} \sum_{i=1}^{n} (\mathcal{E}_i')^2 \; ; \quad P(\mathcal{E}' > \mathcal{E}_o')_A \to P_1(\mathcal{E}' > \mathcal{E}_o')_{A\text{-expt'l}} . \tag{6.8}$$

B. Class B Interference:

Here we have a six-parameter model for the Class B cases. Table (6.2) summarizes the global and generic parameters involved:

Table 6.2: Class B Parameters

Basic I:	Global:	$(A_\alpha, \alpha, A_B, \Gamma'_B, \Omega_{2B}, N_I)$
		$\rightarrow$ [Practical Global: $(A_\alpha, \alpha, A_B, \Gamma'_B, K_B; N_I)$]
Basic II:	Generic:	$(A_B, \alpha, \sigma_G^2, \langle \hat{B}_{oB}^2 \rangle, \langle B_{oB}^\alpha \rangle; N_I)$

(6.9)

The global and generic parameters are related by

$$
\left.\begin{aligned}
\sigma_G^2 &= \Omega_{2B}\Gamma'_B, \text{ [Eq. (3.2a)]} \quad ; \\
&= \sigma_R^2 + \sigma_E^2 , \text{ [Eq. (2.47)]} \quad ; \\
\left\langle \hat{B}_{oB}^2 \right\rangle &= 2\Omega_{2B}\, A_B^{-1} \, , \quad , \text{ [Eq. (3.2a)]} \\
\left\langle \hat{B}_{oB}^\alpha \right\rangle &= (2\Omega_{2B}(1+\Gamma'_B))^{\alpha/2} \frac{\Gamma(1+\alpha/2)A_\alpha}{2\Gamma(1-\alpha/2)A_B} \quad \text{[Eq. (3.12a)]}
\end{aligned}\right\}
$$

and (6.10)

$$\mathcal{E}_B = K_B \mathcal{E}'_B, \quad \text{[cf. (6.11) below]} .$$

The common global and generic parameters are clearly (A_B, α).

The fact that there are six generic parameters for our statistical-physical model of Class B interference stems directly from our pair of approximations $\hat{F}_{1-I}, \hat{F}_{1-II}$, Eqs. (2.90), (2.93), to the exact cf. (2.87): (i), the Impulsive Index A_B [(2.38), (2.39) in (2.51); (ii), the spatial density-propagation parameter α, (2.82); (iii), the independent gaussian component σ_G^2, [(2.47), (2.88c)]; (iv). the α-moment of the generic, filtered envelope waveform $\left\langle \hat{B}_{oB}^\alpha \right\rangle$, cf. (2.87a) , (2.87d); (v), the mean-square of this generic waveform, $\langle \hat{B}_{oB}^2 \rangle$; and finally, (vi), the scaling factor

$N_I(A_\alpha,\alpha,A_B,\Omega_{2B},\Gamma_B'/\mathcal{E}_B)$, cf. remarks in Sec. 3.2-A. This factor N_I is functionally involved with but not solely determined by the other global (and generic parameters, through the APD form, <u>and is independent of</u> $\mathcal{E}_B$ [cf. remarks below in Sec. 6C]; hence it is regarded as a generic parameter here also. The quantity N_I ranges from O(10db) to O(50,60db) in practice. For example, comparison of Fig. (2.4) with Fig. 3.3(II) ($A_\alpha \doteq 1$, $\alpha \doteq 1$), for the same P_1=0.9 gives $N_I \doteq -6-(-44) \doteq 38$db. The point of inflexion, or "bendover" point $\mathcal{E}_B$, cf. Fig. (3.5)II, at which the PD's (and pdf's) corresponding to the two approximating c.f.'s, $\hat{F}_{1-I}$, $\hat{F}_{1-II}$, are joined, to give us the desired composite PD (and pdf), is purely empirical [(vi), (3.18)]. The conversion factor K_B is here

$$K_B \equiv \left\{\hat{\sigma}_G^2/\Omega_{2B}(1+\Gamma_B')\right\}^{1/2} \quad , \tag{6.11}$$

cf. (6.4b), where, again, $\hat{\sigma}_G^2$ is a known (measured) gaussian noise reference level. Also as before, we may obtain the components of σ_G^2 as indicated above, cf. (6.3) et seq. [See, also, footnote, Eq. (6.4a).]

Now to obtain the desired global parameters of our model from observed data we need six convenient nonidentical relations involving these parameters in various, sometimes simple ways. First, we use the exact expression for the mean square envelope (5.14), with the renormalization (6.4), to write

$$\langle(\mathcal{E}')^2\rangle_B = \langle\mathcal{E}^2\rangle_B K_B^{-2} = 1/K_B^2 = \Omega_{2B}(1+\Gamma_B')/\hat{\sigma}_G^2 = \Delta\sigma_G^2/\hat{\sigma}_G^2 \ , \tag{6.12}$$

cf. (6.5). Since the expressions for the Class B moments are analytically quite involved, and because the PD contains all moment information here, we use [(i), (iii), (iv)] of (3.18), (3.19) where P_{1-B} is empirically determined from the data. Accordingly, we have the additional five relations:

<u>rayleigh region</u>: [2 relations]

$$P_1(\mathcal{E}'\geq\mathcal{E}_0')_B \simeq \frac{(\mathcal{E}_0'K_BN_I)^2}{4G_B^2}\sum_{n=0}^{\infty}\frac{(-1)^n\hat{A}_\alpha}{n!}\Gamma(1+\frac{\alpha n}{2})$$

$$\cong (\mathcal{E}_0'K_B)^2\,\frac{e^{-A_B}}{4G_B^2}\sum_{m=0}^{\infty}\frac{A_0^m}{m!}(2\hat{\sigma}_{mB}^2)^{-1} ; \tag{6.13a}$$

large thresholds: [2 relations]

$$P_1(\mathcal{E}' \geq \mathcal{E}'_B)_B \cong \frac{\hat{A}_\alpha \Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\alpha/2)} \left(\frac{\mathcal{E}'_B K_B N_I}{2G_B}\right)^{-\alpha} [1+O_{(iii)}(\mathcal{E}'_B N_I^{-\alpha} K_B^{-\alpha}, \mathcal{E}_B'^{-2} K_B^{-2})]$$

$$\cong \frac{e^{-A_B}}{4G_B^2} \sum_{m=0}^{\infty} \frac{A_B^m}{m!} e^{-(\mathcal{E}'_B K_B)^2/2\hat{\sigma}_{mB}^2} \qquad (6.13b)$$

at bend-over point:

$$\frac{\hat{A}_\alpha \alpha \Gamma(1+\alpha/2)}{\Gamma(1-\alpha/2)} \frac{(2G_B)^{\alpha+1}}{(\mathcal{E}_B N_I)^{\alpha+1}} [1+O(\ )] = \frac{\mathcal{E}_B e^{-A_B}}{4G_B^2} \sum_{m=0}^{\infty} \frac{A_B^m e^{-\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2}}{m!}$$

As noted from (3.18), (v), (vi), $\mathcal{E}_B$ is the joining point (or point of continuity (through the second derivative of $P_{1-I,II}$, at least) for the approximations to P_{1-B}. $\mathcal{E}_B$ is the point of inflexion of P_{1-B}, obtained from $(P_{1-B})_{expt}$. Here

$$\mathcal{E}'_0 = E_0/\sqrt{2\hat{\sigma}_G^2} \ . \qquad (6.14)$$

In practice, $\mathcal{E}'_B$ is available from inspection of the experimental* PD, $P_{1-B,exp}$, so that in addition to (6.12) only the five relations (6.13a,b,c) are then required for the remaining six global parameters. Once these have been obtained, we may use (6.10) to determine the six ultimately generic parameters of the Class B interference under study. Of course, in practice, our data are finite and P_{1-B} is an empirical function; $\langle(\mathcal{E}')^2\rangle_B$ is an estimate, cf. (6.8), based on sample values, and $\mathcal{E}'_B$ is likewise an estimate by inspection, so that all parameters actually obtained are necessarily themselves estimates. We do not include $\mathcal{E}_B$ in our list above of global parameters, and exclude it from the basic, or generic parameters, cf. (6.9), since it is in effect, an empirical quantity resulting from the procedure of joining P_{I-1}, P_{1-II} in approximation to the true P_{1-B}, at $\mathcal{E} = \mathcal{E}_B$.

C. Degenerate Cases:

When $\mathcal{E}_B$ (or $\mathcal{E}'_B$) is not known -- i.e., is not evident from the empiri-

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* However, see the important situation discussed in C following.

cal PD, P_{1-B} -- we can only work with the P_{1-I} form of the PD, namely the approximation suitable for small and intermediate values of $\mathcal{E}_0$ (and $\mathcal{E}_0'$) $< \mathcal{E}_B(\mathcal{E}_B')$. This is the case, for example, of much of the atmospheric noise data, cf. Fig. 2.4, where no bend-over point is at all evident. The model now reduces from six to a five-parameter approximation, in $(A_\alpha, \alpha, \Gamma_B', \Omega_B, N_I)$ [or $(A_\alpha, \alpha, \Gamma_B', K_B, N_I)$] for the global parameters and in $(\alpha, \sigma_G^2, N_I)$ for the generic parameters, cf. Table (6.2). Because $\mathcal{E}_B$ is not known, we are unable to obtain A_B, and hence we can determine only $\Omega_{2B} = A_B\langle \hat{B}_{oB}^2 \rangle/2$, $A_\alpha \sim \langle \hat{B}_{oB}^\alpha \rangle A_B$, and not their individual factors $\langle \hat{B}_{oB}^\alpha \rangle, \langle \hat{B}_{oB}^2 \rangle$. For these five global parameters we need accordingly five equations. The conversion factor K_B is again given by (6.12), and for the four other parameters we use

Eq. (3.11b): rayleigh region:

$$P_1(\mathcal{E}' \geq \mathcal{E}_0')_{B-I} \cong \left(\frac{\mathcal{E}_0' K_B N_I}{2G_B}\right)^2 \sum_{n=0}^{\infty} \frac{(-1)^n \hat{A}_\alpha^n}{n!} \Gamma(1+\frac{\alpha n}{2}) = 0.99, \text{ say}; \tag{6.15a}$$

and

P_{1-I} in the "bend-up" region, where P_{1-I} departs from the "straight line" rayleigh form, so that (3.11b) fully applies, and two points $P_{1-I} = P_3, P_4$ with

Eq. (3.15): large $\mathcal{E}_0'$ $(<\mathcal{E}_B')$:

$$P_1(\mathcal{E}' \geq \mathcal{E}_0')_{B-I} \simeq \frac{\hat{A}_\alpha \Gamma(1+\alpha/2)(\mathcal{E}_B' K_B N_I)^{-\alpha}}{\Gamma(1-\alpha/2)\ (2G_B)^{-\alpha}} (1+O_{(iii)}] = P_3, P_4 \tag{6.15b}$$

where the PD's are empirically determined. Without the turnover point $\mathcal{E}_B$ we cannot join the large-threshold approximation P_{1-BII} to P_{1-BI} for $(\mathcal{E}_0 < \mathcal{E}_B)$, and are thus unable to determine the generic parameters, except for α, σ_G^2, N_I. This indicates the importance of obtaining the "rare-event" data $(\mathcal{E} > \mathcal{E}_B)$, so that the fundamental (i.e., generic) parameters of the interference model may be estimated, as the fundamental descriptors of this noise environment, as specified, of course, by our statistical-physical model in this case [cf. Section (2.1) et seq.].

7. PRACTICAL CONDITIONS FOR CLASS A AND CLASS B INTERFERENCE

In our preceeding analyses we have postulated limiting forms of interference which are strictly Class A or Class B, neglecting the very small contributions of the associated other components (e.g. Class B where Class A is said to occur, etc.). Here we shall establish quantitative conditions which permit us to neglect these other-component effects and to assert that our analytical forms may be applied to the corresponding physical situation, e.g., essentially only Class A, or Class B noise is present.

To do this we start with the relation (2.52) for the general Class C interference and use the results [Eqs. (2.77), (2.78)] for Class A, [Eqs. (2.90), (2.93)] for Class B, to write

$$\hat{F}_1(ir)_{C+G} = \hat{F}_1(ir)_{A+G}\hat{F}_1(ir)_B \tag{7.1a}$$

$$= e^{-A_A}\sum_{m=0}^{\infty}\frac{A_A^m}{m!}e^{-(m/A_A+\Gamma_A')\Omega_{2A}r^2/2} \left\{\begin{array}{ll} e^{-b_{1\alpha}A_B r - b_{2\alpha}A_B r^2/2} & : \ (\hat{F}_{1-I}) \ ; \\ e^{-A_B}\cdot\exp(A_B e^{-b_{2\alpha}r^2/2}) & : \ (\hat{F}_{1-II}) \end{array}\right\}, \tag{7.1b}$$

where we include σ_G^2, (2.47), in the Class A form ($\sim\Gamma_A'$) here.

<u>When Class A noise heavily predominates</u>, we use the transformation $r=a_A\lambda$, cf. (3.3) et seq., and expand the Class B components of the c.f., to get the pair of approximations

$$\hat{F}_1(ia_A\lambda)_{A+G} \doteq e^{-A_A}\sum_{m=0}^{\infty}\frac{A_A^m}{m!}\cdot\left\{\begin{array}{l} e^{-[(m/A_A+\Gamma_A')\Omega_{2A}+b_{2\alpha}A_B]a_A^2\lambda^2/2}\left\{1-b_{1\alpha}A_\alpha a_A^\alpha\lambda^\alpha+O(\lambda^{-\alpha})\right\} \\ \text{and} \\ e^{-[(m/A_A+\Gamma_A')\Omega_{2A}+b_{2\alpha}A_B]a_A^2\lambda^2/2}\left\{1+b_{2\alpha}^2A_B\dfrac{a_A^4\lambda^4}{8}+\ldots\right\} \end{array}\right. . \tag{7.2}$$

For this to reduce to the principal form (3.5), we have at once the <u>two</u> pairs of conditions

<u>Class A:</u>

$$\left.\begin{aligned}
&\frac{b_{2\alpha}A_B}{\Omega_{2A}} = \left(\frac{4-\alpha}{2-\alpha}\right)\frac{\langle\hat{B}^2_{oB}\rangle A_B}{\langle\hat{B}^2_{oA}\rangle A_A}\left(1+2\sigma_G^2/A_A\langle\hat{B}^2_{oA}\rangle\right)^{-1} \ll \Gamma'_A, && (7.3a)\\
&\text{with}\\
&\text{and}\left\{\begin{aligned}
&\frac{b_{1\alpha}A_\alpha}{[2\Omega_A(1+\Gamma'_A)]^{\alpha/2}} = \frac{\Gamma(1-\alpha/2)\langle\hat{B}^\alpha_{oB}\rangle A_B}{2^{\alpha-1}\Gamma(1+\alpha/2)[2\Omega_A(1+\Gamma'_A)]^{\alpha/2}} \ll 1, \quad \text{for } \mathcal{E}<\mathcal{E}_B: && (7.3b)\\
&\frac{b^2_{2\alpha}A_B}{[2\Omega_A(1+\Gamma'_A)]^2} = \left(\frac{4-\alpha}{2-\alpha}\right)^2\frac{\langle\hat{B}^2_{oB}\rangle^2 A_B}{[2\Omega_A(1+\Gamma'_A)]^2} \ll 1, \quad \text{for } \mathcal{E}>\mathcal{E}_B && (7.3c)
\end{aligned}\right.
\end{aligned}\right\}.$$

From (2.37), (2.38), we also note that $A_{A,B}=\gamma_\infty\bar{T}_{s(A,B)}$. Equation (7.3a) is usually the weaker condition and (7.3b,c) the stronger, with the Impulsive Indexes $A_{A,B}$ not too large. A useful, rough rule for considering the interference to be Class A only is, in effect, the condition $A_B \ll A_A$, or $\bar{T}_{sB} \ll \bar{T}_{sA}$, the latter representing the fact that the amount of time, on the average, that the Class A component is present, is much larger than that for the Class B term, a not at all surprising condition from an intuitive viewpoint. The conditions $(\ll 1, \Gamma'_A)$ in (7.3) are, of course, matters of judgment, usually $O(10^{-2}, 10^{-3})$ is sufficient, unless we are concerned with the very rare events, i.e., extremely large values of $\mathcal{E}$. In general, we shall adopt the stricter conditions (7.3b,c). (We shall pursue the detailed anatomy of these and the Class B conditions further in a later study.)

<u>When Class B noise is heavily dominant,</u> on the other hand, we rewrite (7.1b) to include the independent gaussian component embodied in Γ'_A with $\hat{F}_1(ia_B\lambda)_B$, (2.90) and (2.93), so that we have now for the Class A contribution

$$\hat{F}_1(ia_B\lambda)_A \doteq e^{-A_A a_B^2\langle\hat{B}_{oA}^2\rangle\lambda^2/4+O(\lambda^4)} = 1-\frac{A_A\langle\hat{B}_{oA}^2\rangle\lambda^2}{8\Omega_B(1+\Gamma_B)} + O(\lambda^4)\ . \tag{7.4}$$

Accordingly, we get

$$\hat{F}_1(ia_B\lambda)_{B+G} \doteq [\text{Eqs. (2.90) and (2.93)}]\ , \tag{7.5}$$

provided the single condition

Class B:

$$\frac{A_A\langle\hat{B}_{oA}^2\rangle}{8\Omega_{2B}(1+\Gamma_B')} = \frac{1}{4}\left(\frac{2-\alpha}{4-\alpha}\right)\frac{\langle\hat{B}_{oA}^2\rangle A_A}{\langle\hat{B}_{oB}^2\rangle A_B}\cdot\left[1+\sigma_G^2/2\left(\frac{4-\alpha}{2-\alpha}\right)\hat{B}_{oB}^2 A_B\right]^{-1} \ll 1 \tag{7.6}$$

is obeyed. Again, the simple, intuitively obvious condition is that $A_A \ll A_B$ (or $\bar{T}_{sA} \ll \bar{T}_{sB}$): the Class B interference is "on" for a much longer period than the Class A component. The amount of Class A noise is negligible vis-à-vis the Class B contribution. As expected, this condition is, not unexpectedly, just the reverse of that (roughly) required for Class A dominance. [Note, too, that the more precise condition (7.6) is a kind of inverse of conditions (7.3b,c) above for the Class A cases.]

Glossary of Principal Symbols

Equation numbering in Part II does not contain a "II" suffix; when equations from Part II are referred to in Part I, they are so designated, e.g. Eq. (3.5), Part II, etc.

Figure numbering in Part II is similarly indicated by a "II" following the number.

A.	A_o	= peak amplitude of typical input signal
	$A_A, A_B, A_\infty, A_{\infty,A}, A_{\infty,B}$	= Impulsive Indexes, (Class A,B interference)
	A_α	= effective Impulsive Index
	a_A, a_B, a	= normalizing factors
	APD	= a posteriori probability; here 1-Distribution = P_1
	ARI	= combined aperture-IF-IF receiver input stages
	$\mathcal{a}_T, \mathcal{a}_R$	= source, receiver beam patterns
	α	= spatial density-propagation parameter
B.	$B_o, \hat{B}_{oA}, \hat{B}_{oB}$	= generic or typical envelope of waveform from ARI receiver stage
	$b_{1\alpha}, b_{2\alpha}, b_{2\ell+2\|\alpha}$	= weighted moments of the generic envelope B_{oB}
	β	= exponent of moment
C.	c.f.	= characteristic function
D.	D_1	= probability distribution
	δ	= delta (singular) function
E.	E, E_o	= instantaneous envelope
	$e_{o\gamma}$	= limiting receiver voltage
	$\mathcal{E}, \mathcal{E}_o, \mathcal{E}_0', \mathcal{E}_o', \hat{\mathcal{E}}_o$	= normalized (instantaneous) envelopes; ϵ_o=envelope threshold
	$\mathcal{E}_B$	= "bend-over" point (Class B), empirical pt. of inflexion in P_{1-B}

	η	= an exponent
	$\hat{\varepsilon}$	= impulse epoch
	$\varepsilon_o, \varepsilon_d$	= normalized doppler
F.	$\hat{F}_1, F_1$	= characteristic functions
	${}_1F_1$	= confluent hypergeometric function
	$\Delta f_N, \Delta f_{ARI}$	= noise, receiver bandwidths
	f	= frequency
G.	G_o	= a basic waveform
	$g(\lambda)$	= geometrical factor of received waveform
	Γ'_A, Γ'_B	= ratio of (intensity of) gaussian component to that of the "impulsive", or nongaussian component
	$\Gamma(x)$	= gamma function
	γ	= exponent of propagation law, with range
H.		
I.	$\hat{I}_T, \hat{I}_\infty$	= exponent of characteristic function
	I_c	= incomplete Γ-function
	$\hat{i}_R$	= unit vector
J.	J_o, J_1	= Bessel function, 1st-kind, (0,1 order).
	J_Λ	= jacobian
K.	$K_{A,B}$	= conversion factor, for arbitrary normalization
L.	Λ	= domain of integration
	λ	= argument of the c.f.
	$\underset{\sim}{\lambda}$	= (λ, θ, ϕ), coordinates
M.	μ	= exponent of source density law with range
	μ_d	= normalized doppler
N.	n.b.	= narrow-band

O.	Ω_{2A},Ω_{2B}	= mean intensity of the nongaussian component
	ω,ω_o	= angular frequencies (ω_o=carrier angular fr.)
P.	P_1	= APD or exceedance probability
	pdf	= probability density function
	Ψ,ϕ	= phase of narrow band wave
	ϕ_T,ϕ_R	= aperture phase
Q.		
R.	r	= c.f. variable
	ρ	= poisson "density"
S.	$\sigma,\sigma_G,\hat{\sigma},\hat{\sigma}_{mA,B},\Delta\sigma_G^2,\sigma_\Lambda,\sigma_R^2,\sigma_E^2$	= variances
	$\sigma_{S,V}$	= source density
T.	$T_s,\bar{T}_{s;A,B}$	= emission duration
	$t,t_1,t_2,$	= times
	θ,θ'	= sets of waveform parameters
U.	U,U_{nb}	= basic waveforms out of ARI receiver stage
	$u_o,u_{oA,B}$	= normalized envelope waveform at output of ARI ARI stages
V.		
W.	w_1,W_1	= probability density function
X.	X	= instantaneous amplitude
	x_o	= a c.f. variable
Y.		
Z.	z_o	= a normalized time

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15. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

Most man-made and natural electromagnetic interferences are highly non-gaussian random processes, whose degrading effects on system performance can be severe, particularly on most conventional systems, which are designed for optimal or near optimal performance against normal noise. In addition, the nature, origins, measurement and prediction of the general EM interference environment are a major concern of an adequate spectral management program. Accordingly, this second study in a continuing series [cf., Middleton, 1974] is devoted to the development of analytically tractable, experimentally verifiable, statistical-physical models of such electromagnetic interference.

Here, classification into three major types of noise is made: Class A (narrowband vis-à-vis the receiver), Class B (broadband vis-à-vis the receiver), and Class C (=Class A + Class B). First-order statistical models are constructed for the Class A and Class B cases. In particular, the APD (a posteriori probability distribution) or exceedance probability, PD, viz. $P_1(\mathcal{E} > \mathcal{E}_0)_{A,B}$, and the

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associated probability densities, pdf's, $w_1(\hat{\mathcal{E}})_{A,B}$, of the envelope are obtained; [the phase is shown to be uniformly distributed $(0,2\pi)$]. These results are canonical, i.e., their analytic forms are invariant of the particular noise source and its quantifying parameter values, levels, etc. Class A interference is described by a 3-parameter model, Class B noise by a 6-parameter model. All parameters are deducible from measurement, and like the APD's and pdf's, are also canonical in form: their structure is based on the general physics underlying the propagation and reception processes involved, and they, too, are invariant with respect to form and occurrence of particular interference sources.

Excellent agreement between theory and experiment is demonstrated, for many types of EM noise, man-made and natural, as shown by a broad spectrum of examples. Results for the moments of these distribution are included, and more precise analytical conditions for distinguishing between Class A,B, and C interference are also given. Methods for estimating the canonical model parameters from experimental data (essentially embodied in the APD) are outlined in some detail, and a program of possible next steps in developing the theory of these highly nongaussian random processes for application to general problems of spectrum management is presented.

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