NTIA Technical Memorandum TM-05-423

Analysis of the Markov Character of a General Rayleigh Fading Channel

Roger Dalke
George Hufford

U.S. DEPARTMENT OF COMMERCE
Carlos M. Gutierrez, Secretary

Michael D. Gallagher, Assistant Secretary
for Communications and Information

April 2005

CONTENTS

ANALYSIS OF THE MARKOV CHARACTER OF A GENERAL RAYLEIGH FADING CHANNEL

Roger Dalke and George Hufford*

It has been proposed that first-order Markov channel models can be used to adequately predict the behavior of a mobile "Rayleigh" fading channel and hence improve the reliability of bidirectional mobile communications systems. Previous authors have addressed this question by applying information theory to the amplitude statistics of a stationary mobile communications channel. The previous work required numerical analysis to show that for a particular covariance function and range of relevant parameters (i.e., Doppler frequency, symbol period), the channel is approximately first-order Markov. In our analysis, both amplitude and phase information are used to obtain analytic expressions which can easily be used to determine if a non-stationary arbitrary Rayleigh channel is necessarily first-order Markov. The analytic results are given in terms of arbitrary covariance functions that can readily be applied to measurements. In particular, our results show that the previously studied mobile channel is not first-order Markov in character.

Key words: Gaussian process; information theory; Markov process; mobile communications; Rayleigh fading; reliability

1. INTRODUCTION

The performance of networks using wireless channels can be significantly affected by channel impairments such as multipath fading. Various techniques have been proposed to mitigate the fading problem. An important example is the mobile channel which is characterized by slow Rayleigh fading. Improved reliability in bidirectional wireless communications networks utilizing mobile channels can be achieved by employing forward-error-correction (FEC) combined with automatic-repeat-request (ARQ) hybrid protocols [1]. Such techniques require channel models that can be used to predict fading characteristics of the channel. In practice, this is accomplished by assuming a finite-state first-order Markov channel model based on the assumption that current information about the state of the channel (e.g., signal-to-noise ratio) can be used to reliably predict the state of the channel in the future (i.e., when the next symbol is received) [2].

Information theoretic concepts of entropy and mutual information have been used to demonstrate the validity of the first-order Markov assumption for a Rayleigh fading channel [2]. In

*The authors are with the Institute for Telecommunication Sciences, National Telecommunications and Information Administration, U. S. Department of Commerce, Boulder, Colorado 80305.

that work, the theoretical mobile channel described by Jakes [3] was used to calculate the mutual information relating the channel amplitude for a particular symbol to the channel amplitude at the time of the previous two symbols. The resulting mathematical expressions are quite complicated, requiring numerical analysis and simulations to demonstrate the authors' assertion that the first-order Markov model is valid. The results presented are not general, in that this mobile channel is stationary and has a particular autocorrelation function.

In what follows, the same information theoretic concepts are used to analyze the Markov character of an arbitrary nonstationary Rayleigh process (i.e., the autocorrelation function is not specified). The advantage of the approach presented in this paper is that it yields straightforward analytic expressions that can be applied to arbitrary Rayleigh channels and can be used in conjunction with channel measurements. As an example, the results of this analysis are applied to the mobile channel described in [3]. It is found that contrary to the results given in [2], this particular channel is not approximately Markov.

2. ANALYSIS

For the purposes of this analysis, it is assumed that the communication system determines the Rayleigh channel's amplitude and phase at a sequence of discrete times and uses this information to predict the channel's behavior one period into the future. This can be accomplished, for example, by the use of pilot tones as described in [4]. At baseband, the channel amplitude and phase at the time of the kth period are represented as a zero mean normally distributed complex stochastic process X_k. The mutual information relating the kth channel value to the previous two values is given by the relative entropy between the joint distribution and the product of the marginal distributions [5]:

$$I(X_k; X_{k-1}, X_{k-2}) = \mathcal{E}\left\{\log_2 \frac{f(X_{k-2}, X_{k-1}, X_k)}{f(\cdot, \cdot, X_k) f(X_{k-2}, X_{k-1}, \cdot)}\right\} \tag{1}$$

where $\mathcal{E}$ denotes expectation, f is the joint density function and the dots in place of arguments indicate marginal distributions in the remaining variables. Following [2], the first-order Markov approximation is considered valid if the mutual information relating X_k and X_{k-1} is approximately the same as the mutual information relating X_k and the combination of both X_{k-2} and X_{k-1}. More explicitly, using the chain rule for mutual information [5], we wish to determine conditions for which

$$I(X_k; X_{k-1}, X_{k-2}) - I(X_k; X_{k-1}) = I(X_k; X_{k-2}|X_{k-1}) \approx 0 \tag{2}$$

where

$$I(X_k; X_{k-2}|X_{k-1}) = \mathcal{E}\left\{\log_2 \frac{f(X_{k-2}, X_k|X_{k-1})}{f(X_{k-2}, \cdot|X_{k-1}) f(\cdot, X_k|X_{k-1})}\right\} \tag{3}$$

and the conditional densities have the usual meaning, e.g.,

$$f(x_{k-2}, x_k|x_{k-1}) = \frac{f(x_{k-2}, x_{k-1}, x_k)}{f(\cdot, x_{k-1}, \cdot)}. \tag{4}$$

2.1 Mutual Information for Complex Normal Multivariates

To proceed further, it is useful to obtain general expressions for the mutual information of complex Gaussian multivariate processes. Let U and V be complex multivariate random vectors of dimension n and m with normally distributed zero mean components. The mutual information $I(U; V)$ is obtained by extending the results presented in [6] to complex normal variates as outlined below.

Let $Z = (U, V)$ and W be the $(m+n) \times (m+n)$ covariance matrix with components $\mathcal{E}\{Z_i Z_j^*\}$. The asterisk $*$ denotes complex conjugate (and Hermitian transpose in the case of matrices). W can be written as a partitioned matrix

$$W = \begin{pmatrix} W_u & S^* \\ S & W_v \end{pmatrix} \tag{5}$$

where W_u has components $\mathcal{E}\{U_i U_j^*\}$ and W_v has components $\mathcal{E}\{V_i V_j^*\}$. The density functions required to calculate the mutual information are

$$f(u,v) = \frac{1}{\pi^{n+m}\det W} e^{-z^* W^{-1} z} \tag{6}$$
$$f(u,\cdot) = \frac{1}{\pi^{n}\det W_u} e^{-u^* W_u^{-1} u} \tag{7}$$
$$f(\cdot,v) = \frac{1}{\pi^{m}\det W_v} e^{-v^* W_v^{-1} v} \tag{8}$$

hence,

$$\begin{aligned} I(U;V) &= \mathcal{E}\left\{\log_2 \frac{f(U,V)}{f(U,\cdot)f(\cdot,V)}\right\} \\ &= \log_2\left(\frac{\det W_u \det W_v}{\det W}\right) \\ &\quad - \mathcal{E}\{Z^* W^{-1} Z\} + \mathcal{E}\{U^* W_u^{-1} U\} + \mathcal{E}\{V^* W_v^{-1} V\}. \end{aligned} \tag{9}$$

For any n-dimensional complex multivariate vector Z and any $n \times n$ matrix A,

$$\begin{aligned} \mathcal{E}\{Z^* A Z\} &= \sum_{i,j} A_{ij}\mathcal{E}\{Z_i^* Z_j\} = \sum_{i,j} A_{ij} W_{ji} \\ &= \mathrm{Trace}\{AW\} \end{aligned} \tag{10}$$

and consequently

$$\mathcal{E}\{Z^* W^{-1} Z\} - \mathcal{E}\{U^* W_u^{-1} U\} - \mathcal{E}\{V^* W_v^{-1} V\} = 0 \tag{11}$$

whence,

$$I(U;V) = \log_2\left(\frac{\det W_u \det W_v}{\det W}\right). \tag{12}$$

2.2 Conditional Mutual Information

In this section, we use the results of the previous section to obtain an expression for the conditional mutual information given in Equation 3. Without loss of generality, we simplify the notation by setting the index $k = 3$ with the caveat that in general the parameters of the process are a function of time. Using the results from the previous section, let $U = X_3$ and $V = (X_2, X_1)$, then

$$I(X_3; X_2, X_1) = \log_2 \frac{\det W_u \det W_v}{\det W} \tag{13}$$

and setting $\hat{V} = X_2$, $\hat{Z} = (U, \hat{V})$, and $\hat{W}_{ij} = \mathcal{E}\{\hat{Z}_i \hat{Z}_j^*\}$ we obtain

$$I(X_3; X_2) = \log_2 \frac{\det W_u \det W_{\hat{v}}}{\det \hat{W}}. \tag{14}$$

The desired conditional mutual information is

$$I(X_3; X_1|X_2) = \log_2 \frac{\det W_v \det \hat{W}}{\det W_{\hat{v}} \det W}. \tag{15}$$

At this point it is useful to adopt the following notation: $\sigma_k^2 = \mathcal{E}\{|X_k|^2\}$ and $\sigma_i\sigma_j\rho_{ij} = \mathcal{E}\{X_i X_j^*\}$ where, of course, the σ_i are real and $\rho_{ji} = \rho_{ij}^*$. Then

$$\begin{aligned} \det W &= \sigma_1^2\sigma_2^2\sigma_3^2(1 + 2\Re\{\rho_{12}\rho_{23}\rho_{31}\} - |\rho_{12}|^2 - |\rho_{23}|^2 - |\rho_{13}|^2) \\ &= \sigma_1^2\sigma_2^2\sigma_3^2[(1 - |\rho_{12}|^2)(1 - |\rho_{23}|^2) - |\rho_{13} - \rho_{12}\rho_{23}|^2] \end{aligned} \tag{16}$$

$$\det \hat{W} = \sigma_2^2\sigma_3^2(1 - |\rho_{23}|^2) \tag{17}$$

$$\det W_v = \sigma_1^2\sigma_2^2(1 - |\rho_{12}|^2) \tag{18}$$

$$\det W_{\hat{v}} = \sigma_2^2. \tag{19}$$

The conditional mutual information can be written as

$$I(X_3; X_1|X_2) = \log_2 \frac{(1 - |\rho_{12}|^2)(1 - |\rho_{23}|^2)}{(1 - |\rho_{12}|^2)(1 - |\rho_{23}|^2) - |\rho_{13} - \rho_{12}\rho_{23}|^2}. \tag{20}$$

A further simplification is achieved by defining

$$\gamma^2 = \frac{|\rho_{13} - \rho_{12}\rho_{23}|^2}{(1 - |\rho_{12}|^2)(1 - |\rho_{23}|^2)} \tag{21}$$

yielding

$$I(X_3; X_1|X_2) = -\log_2(1 - \gamma^2). \tag{22}$$

Clearly, the quantity γ^2 must satisfy $0 \leq \gamma^2 < 1$, and note what happens if the process is Markov. In that case the Chapman-Kolmogorov equation holds and may be written as

$$f(x_3|x_1) = \iint_{-\infty}^{\infty} f(x_3|x_2) f(x_2|x_1)\, dx_{r2} dx_{i2} \tag{23}$$

(where x_{r2} and x_{i2} are the real and imaginary parts). From this there follows, for example, that $\mathcal{E}\{X_3|X_1\} = \mathcal{E}\{\mathcal{E}\{X_3|X_2\}|X_1\}$. But the conditional density functions, such as $f(x_3|x_1)$, are again Gaussian with, however, new means and variances. The mean turns out to be, for example, $(\sigma_1/\sigma_3)\rho_{31}X_1$, and we quickly find $\rho_{13} = \rho_{12}\rho_{23}$ and thus that $\gamma^2 = 0$. One expects this, for then also $I(X_3; X_1|X_2) = 0$—if one knows X_2 then the value of X_1 contributes no information towards estimating X_3.

2.3 The Nearly Markov Condition

Following [2, 7], we would say that a process is "nearly Markov" if the ratio

$$M = \frac{I(X_3; X_1|X_2)}{I(X_3; X_2, X_1)} \ll 1. \tag{24}$$

Using the results from the previous section,

$$I(X_3; X_2, X_1) = -\log_2[(1 - |\rho_{23}|^2)(1 - \gamma^2)] \tag{25}$$

whence

$$M = \frac{-\log_2(1 - \gamma^2)}{-\log_2[(1 - |\rho_{23}|^2)(1 - \gamma^2)]}. \tag{26}$$

We note that $0 \leq M \leq 1$, and that when M is nearly 0 then γ must be much smaller than $|\rho_{23}|$. In some sense the ratio M is a normalized way to say $\rho_{13} \approx \rho_{12}\rho_{23}$.

Conversely, if the process is nearly Markov, then γ^2 is nearly 0 and, provided ρ_{23} is not also small, there would follow

$$M \approx \frac{\gamma^2}{-\log_2(1 - |\rho_{23}|^2)} \tag{27}$$

which ought also to be small.

When the process is stationary, the correlations become one dimensional so that we could write $\rho_{jk} = \rho_{k-j}$. Our results are simplified a little bit and we find $\rho_{12} = \rho_{23} = \rho_1$ and $\rho_{13} = \rho_2$ and

$$\gamma^2 = \frac{|\rho_2 - \rho_1^2|^2}{(1 - |\rho_1|^2)^2} \qquad M = \frac{-\log_2(1 - \gamma^2)}{-\log_2[(1 - |\rho_1|^2)(1 - \gamma^2)]}. \tag{28}$$

2.4 An Example—The Mobile Channel

As our example, consider the mobile channel described by [3] and analyzed in [2]. The channel is stationary and the correlations are given by $\rho_k = J_0(k\omega_m\tau)$ where $\omega_m = 2\pi v/\lambda$ is the maximum Doppler frequency, and where v is the speed of the mobile vehicle, λ is the wavelength of the radio transmission, τ is the sample period, and J_0 is the zero-order Bessel function. The idea suggested by [2] is to sample at every symbol. This would mean that τ is small. Indeed, the authors of [2] would say that $.0002 < \omega_m\tau < .004$; and if this is true then the Bessel functions can be reasonably approximated as

$$J_0(z) \approx 1 - \frac{z^2}{4} + \frac{z^4}{64}.$$

Then from Equation 28 we find

$$\begin{aligned} \gamma^2 &\approx \frac{\frac{1}{4}(\omega_m\tau)^4(1 - \frac{5}{8}(\omega_m\tau)^2)}{\frac{1}{4}(\omega_m\tau)^4(1 - \frac{3}{8}(\omega_m\tau)^2)} \approx 1 - \frac{1}{4}(\omega_m\tau)^2 \\ M &\approx \frac{-\log_2[(\omega_m\tau)^2/4]}{-\log_2[(\omega_m\tau)^4/8]} \approx \frac{1}{2}\frac{-\log_2\omega_m\tau + 1}{-\log_2\omega_m\tau + 3/4} \approx \frac{1}{2} \end{aligned} \tag{29}$$

This result indicates that the channel is not approximately Markov. In fact, only about half the mutual information $I(X_3; X_1, X_2)$ is obtained from the previous sample X_2 alone. This conclusion is not consistent with the result given in [2].

3. DISCUSSION AND CONCLUSIONS

The importance of predicting the future state of a Rayleigh fading channel based on present channel conditions has been described in the literature (e.g., [2, 3, 7, 8]). In essence, such knowledge can be used in conjunction with error-control coding and feedback systems to improve the reliability of mobile communications systems. Owing to their simplicity, it is desirable to obtain and use finite-state first-order Markov models to predict the state of the channel one symbol period into the future. To this end, technical papers have been published [2, 7] that "verify" the validity of the first-order Markov assumption for a mobile communications channel. From our point of view, this analysis seemed to be quite limited in that the approach used required numerical analysis techniques to calculate the mutual information. In addition, only channel amplitudes were used to calculate relative entropy. As a consequence, the results are limited to the particular channel that was analyzed and cannot be applied to more general nonstationary Rayleigh channels with arbitrary covariance functions.

In studying this problem, we found that by considering both the amplitude and the phase of a general Rayleigh fading channel it was in fact possible to obtain analytic expressions describing the efficacy of the first-order Markov assumption. The results of this analysis are given in Section 2.2. The advantage of this approach is that the results can readily be applied to nonstationary channels with arbitrary covariance functions. The only assumption used in our analysis is that the fading is assumed to be approximately frequency flat with Rayleigh amplitude distributions such as may be found in a typical mobile communications environment. The primary value of our analysis is that it can easily be used to characterize a particular mobile communications environment.

As an example, the results of our analysis were applied to the mobile channel used in [2]. It is worthwhile to note that our results contradict the previous conclusion [2] that the mobile channel is approximately first-order Markov in character. Specifically, we found that only about half of the available information was obtained using just one channel state. This result may be understood heuristically by considering a slowly varying Rayleigh channel with a particular trend (i.e., the channel is fading, coming out of a fade, or relatively flat) at the time of interest. If one considers the channel state at a particular symbol reception time and uses it to predict the channel state one symbol period into the future without any additional information, the best guess is to perhaps assume that the future state will be about the same as the current state. Knowing only one state, it is not possible to predict which trend is most likely. It seems obvious that given an additional past state one can better predict the trend and hence obtain a better estimate of the future state. This is consistent with what was found in the case of the mobile channel where, considering the total information available about a future state based on the current and a previous state, only about half the information is available from knowing the current state alone.

4. REFERENCES

[1] B. S. Wicker, "Reed-Soloman error control coding for Rayleigh fading channels with feedback," *IEEE Trans. Vehicular Tech.*, vol. VT-41, no. 2, pp. 124–133, May 1992.

[2] H. S. Wang and P. Chang, "On verifying the first-order Markovian assumption for a Rayleigh fading channel model," *IEEE Trans. Vehicular Tech.*, vol VT-45, no.2, pp. 353–357, May 1996.

[3] W. C. Jakes, ed., *Microwave Mobile Communications*, ch.1, New York: Wiley, 1974.

[4] F. Davarian, "Mobile digital communications via tone calibration," *IEEE Trans. Vehicular Tech.*, vol VT-36, pp. 55–62, May 1987.

[5] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, ch. 2, New York: Wiley, 1991.

[6] S. Kullback, *Information Theory and Statistics*, ch. 9, New York: Wiley, 1959.

[7] H. S. Wang and N. Moayeri, "Finite-state Markov channel—A useful model for radio communication channels," *IEEE Trans. Vehicular Tech.*, vol. VT-44, pp. 163–171, Feb. 1995.

[8] M. Zorzi, R. R. Rao, and L. B. Milstein, "Error statistics in data transmission over fading channels," *IEEE Trans. Communications*, vol. COM-46, no. 11, pp. 1468–1477, Nov. 1998.