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Relationships between Gilbert-Elliot Burst Error Model Parameters and Error Statistics

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Technical Memorandum

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RELATIONSHIPS BETWEEN GILBERT-ELLIOT BURST ERROR MODEL PARAMETERS AND ERROR STATISTICS

Jaden Pieper, Stephen Voran[1]

The Gilbert-Elliot model is a popular and effective tool for modeling bursty (non-independent) errors in communication links. This memorandum provides linkages between model parameters and error statistics. The motivation is that these linkages can allow users to control models in order to obtain desired error statistics without any detailed understanding of Markov chains or probability. Features such as error rate and expected burst length are intuitive and also directly measurable in an error stream. This makes them natural candidates for controlling models after they are converted to the necessary model parameters (probabilities). We consider three different versions of the Gilbert-Elliot model and we present results for each. We also describe software that can be used to convert between error statistics and model parameters, to generate error patterns from a variety of variables, and also to estimate model parameters from an input error stream. This software is available at `https://doi.org/10.5281/zenodo.7438482`

Keywords: bit-errors, bursty errors, error statistics, Gilbert-Elliot, Markov chain, packet-loss, software simulation

1 INTRODUCTION

In practice, most digital communication links are imperfect—bits may be inverted and data packets may be "lost" or delayed sufficiently that they must be declared to be "lost." Modeling and simulating these imperfections or errors are common and important problems. A simple and intuitive approach would be to set an error rate and simulate errors that are independent. In other words, the probability of seeing an error is completely independent from whether or not an error just occurred. A reasonable next step is to add a layer of complexity to better mimic reality by including the concept of burstiness—the physics of many communications links cause errors to occur in bursts rather than independently. The Gilbert-Elliot model [1] is a Markov chain that provides a simple way to generate error patterns that can be either bursty or independent. The error patterns produced by the model can be used to simulate bit errors or packet losses.

This memorandum equips the reader to successfully control three different variations of the Gilbert-Elliot model to obtain the desired error statistics without relying on an understanding of Markov chains or probability theory. In particular it allows readers to convert intuitive and meaningful target error statistics into the probability values (or model controls) the Gilbert-Elliot model needs in order to generate errors with those desired statistics.

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1.1 The Gilbert-Elliot Model

The Gilbert-Elliot model is a two-state model for describing errors in a digital communication link. These errors define a relationship between a transmitted binary message and the corresponding received binary message. Given a transmitted message of length N, $\{M_n\}_{n=0}^{N-1}$, the received message is given by

$$R_n = M_n + X_n, \tag{1}$$

where X_n describes the error for the nth element of the received message, M_n, R_n, X_n are either 0 or 1, and the addition is modulo two. If the messages are sequences of bits, we say that the modulo-two addition does nothing when the error value is zero, but it flips a bit when the error value is one. Equation (1) can also describe the packet loss process where a one indicates an intact packet and a zero indicates a lost or corrupted packet. Then if $\{M_n\}_{n=0}^{N-1}$ describes a string of intact transmitted data packets ($M_n = 1$ for $n = 1, 2, \ldots, N-1$), the error value zero leaves a packet intact, but the error value one transforms an intact packet to a lost or corrupted packet.

The Gilbert-Elliot model provides a method for generating the errors, X_n which can represent bit errors or packet losses. Examples of the bit-error application can be found in [2]–[5] and examples of the packet-loss application can be found in [6]–[9] . The model can treat bursty errors where the probability of seeing an error is dependent on whether or not an error just occurred.

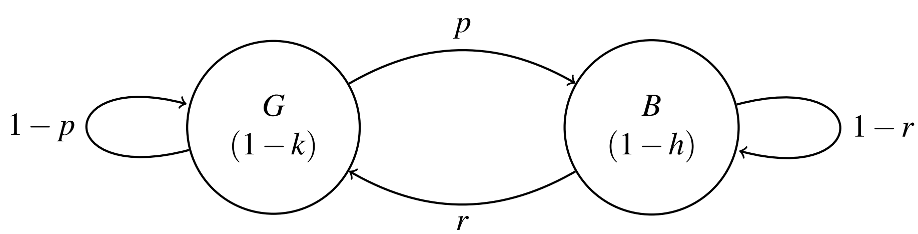


Figure 1: Gilbert-Elliot model.

The model consists of a two-state Markov chain where a state G describes a "good" state and B describes a "bad" or "burst" state, as seen in Figure 1. There are two transition probabilities, $0 < p < 1$ and $0 < r < 1$, that describe the probability of transitioning between the states G and B, where the probability of transitioning from G to B is given by p and the reverse by r. (The cases of p or r being zero or one are mathematically valid but they do not result in useful stochastic models. The value zero causes the model to stay in a single state and the value one forces deterministic state transitions.) Note that the special case $p = 1-r$ gives independent errors—the probability of an error occurring is completely independent of the state of the Markov chain. It follows that $p \neq 1-r$ will give rise to dependent or bursty errors.

There are two additional parameters, $0 \leq k \leq 1$ and $0 \leq h \leq 1$, that describe the probability of no errors occurring in states G and B respectively. The constraint $h < k$ ensures the probability of errors in state B is greater than in state G, and preserves their respective interpretations. These parameters allow for additional realism and complexity in the model; however, fixing these parameters to specific values produces simpler but still interesting models.

In particular, the simplest model consists of just two free parameters (p and r) and occurs when $k = 1$ and $h = 0$. In this case, state G never produces an error and state B always produces errors. This simple model allows for errors to arrive in bursts and is effective for many situations.

Another variation of interest has three free parameters. Here p, r, and h are free but $k = 1$. In this case, state G still cannot produce an error, but the probability of an error occurring when in state B is now $1 - h$. This version can describe physical scenarios where the good state is very good indeed, and the bad state can produce errors but does not always do so.

Finally, when all four parameters are free there is also a probability $(1 - k)$ of an error occurring while in state G. This is the most realistic version of the model, as in any practical communication link there is always some probability of an error occurring, even under ideal conditions. In other words, errors can come from two distinct sources with potentially very different error rates. The parameter k allows for this to be captured.

Whenever one is selecting a model, consideration should be given to the balance between realism and complexity. For some problems the most realistic model may not be worth the additional complexities demanded by it. And some datasets of error patterns might not justify modeling with four, or even three, free parameters. In many instances the two-parameter and three-parameter Gilbert-Elliot model may be sufficient.

Next we provide mappings between selected relevant error statistics and the model parameters for the two-parameter, three-parameter, and four-parameter variations of the model. We can think of these error statistics as "control knobs" or "model controls" that we set so that the model will produce error patterns with those statistics. But to get this result those model control values must first be translated to the probabilities p, r, h, and k that the models require in order to operate.

1.2 Error Statistics of Interest

The Markov model parameters are fairly interpretable, but they do rely on a familiarity with conditional probability. In addition, they do not directly describe features of the error sequences emitted by the model. An engineer interested in using these models to generate realistic error patterns for a given scenario would likely prefer to control the model using error statistics that are directly descriptive of the desired output. Here we will focus on error statistics such as the error rate, the expected length of error bursts, and the proportion of time spent in the bad state, B. We will provide mappings in both directions between these error statistics and the Gilbert-Elliot model parameters.

Here we will explicitly define these error statistics for the four-parameter model. By setting $k = 1$ we can address the three-parameter model. Setting $k = 1$ and $h = 0$ will give the results for the two-parameter model.

Let Z_n describe the state of the Gilbert-Elliot model at time n, so that $Z_n = G$ or $Z_n = B$. Let X_n describe the error at time n, where $X_n = 0$ means no error occurred, and $X_n = 1$ means an error was applied to the received message. We first define the proportion of time spent in the good and bad states, π_G and π_B, which are found by solving for the steady state distribution of the Markov chain in Figure 1. At steady state $\pi_G = (1-p)\pi_G + r(1-\pi_G)$ and this yields

$$\pi_G = \frac{r}{p+r}, \quad \pi_B = 1 - \pi_G = \frac{p}{p+r}, \; 0 < \pi_G, \pi_B, < 1. \tag{2}$$

The error rate will then be the sum across both states of the probability of an error occurring in a state times the proportion of time spent in that state, or

$$\begin{aligned} \bar{x} &= (1-k)\pi_G + (1-h)\pi_B \\ &= \frac{(1-k)r + (1-h)p}{p+r}, \; 0 < \bar{x} < 1. \end{aligned} \tag{3}$$

An error burst is a sequence where $X_n = 1$ for one or more consecutive values of n. An error burst can have length 1, 2, 3, etc. Let L describe the length of an error burst. L is a geometrically distributed random variable with parameter $P(X_n = 0 | X_{n-1} = 1)$. The mean value of a geometrically distributed random variable is the reciprocal of the distribution parameter. So the expected error burst length, L_1, is

$$\begin{aligned} L_1 &= \frac{1}{P(X_n = 0 | X_{n-1} = 1)} \\ &= \frac{(1-k)r + (1-h)p}{(1-k)r((1-p)k + ph) + (1-h)p((1-r)h + rk)}, \; 1 < L_1 < \infty. \end{aligned} \tag{4}$$

Note that the above allows error bursts with infinite length. This greatly simplifies the value of L_1 and is reasonable in practice as the number of bits or packets considered when testing a communications system is very large. The derivation of (4) along with mathematical justification for allowing bursts of infinite length is given in Appendix B.

1.3 Example: Bit Errors in Radio Channels

As mentioned above, the model is very useful for describing bit errors or packet losses. Bit errors in radio channels provide a particularly intuitive example. When the received signal is low enough relative to noise and interference, bit errors become inevitable. The bit-error rate and the temporal distribution of these errors will depend on the specifics of the situation and we offer several examples here.

If the received signal is simply disappearing into the noise floor of a stationary receiver (as it would at the extreme edge of a coverage area) bit errors may occur independently. This can be modeled by the two-parameter model (achieved by setting $k = 1$ so the good state never gives a bit error and $h = 0$ so the bad state always gives a bit error) with the transition probability constraint $p = 1 - r$ (probability of entering bad state does not depend on current state). Under these constraints, the bit error rate is p, so larger values of p will model weaker signals, and smaller values of p will model stronger signals.

If transmitter or receiver or both are moving, the physical path of the radio channel will be changing, and the received signal strength, and the strength of multi-path reflections and other interferers will also be changing. Now the three- or four-parameter model is needed. For example, the good state and the associated bit-error rate $1-k$ can represent the times when the receiver is noise-limited and the error rate is lower, while the bad state and its bit-error rate $1-h$ can represent the times when receiver location causes additional attenuation of the desired signal—perhaps a lower elevation or behind a building. The transition probabilities p and r can then be set to achieve the desired residency in each state (π_G and π_B), or to produce an appropriate average error burst length L_1.

If bursts of bit-errors are due to receiver motion, then faster motion may cause shorter residency in each state, and this can lead to shorter but more frequent bursts. Thus p and r may be adjusted to produce values of L_1 that match receiver motion. Finally, note that even a stationary transmitter and receiver may experience bursty bit errors. Interference is rarely uniform over time, and other moving objects can cause levels of multi-path interference to vary as well. We have provided examples of just a few of the nearly limitless number of scenarios whereby radio channels produce bit errors and we suggest that carefully chosen and tuned Gilbert-Elliot models can sufficiently model a great number of these scenarios.

2 TWO-PARAMETER MODEL

In the two-parameter model state G never gives an error (because $k = 1$) and state B always gives an error (because $h = 0$). Equation (3) shows that in this case $\bar{x} = \pi_B$. That is, the error rate is also the proportion of time spent in the bad state. The two-parameter model has two degrees of freedom and is thus fully specified by two values. Table 1 lists four pairs of values that may be useful and meaningful for controlling the two-parameter model.

Table 1: Sets of model parameters or error statistics for use with the two-parameter model.

Parameters	**Interpretation**	**Equations that convert to** $(p,\ r)$
$(p,\ r)$	Model state transition probabilities	—
$(\bar{x},\ L_1)$	Error rate, expected burst length	(7), (8)
$(\bar{x},\ L_R)$	Error rate, relative expected burst length	(14), (13)
$(\bar{x},\ \gamma)$	Error rate, lag-one error correlation	(18), (19)

2.1 Error Rate and Expected Burst Length

We will first consider using the error rate, $\bar{x}$, and the expected burst length, L_1, to control the model. For the two-parameter case we evaluate (3) at $k = 1$, $h = 0$ to find

$$\bar{x}(p,r) = \frac{p}{p+r}. \tag{5}$$

Similarly, evaluation of (4) gives

$$L_1(p,r) = \frac{1}{r}. \tag{6}$$

It is reasonably easy to invert (5) and (6) to find the model transition probabilities:

$$p(\bar{x},L_1) = \frac{\bar{x}}{L_1 \cdot (1-\bar{x})} \tag{7}$$

and

$$r(\bar{x},L_1) = \frac{1}{L_1}. \tag{8}$$

As both p and r are probabilities the restrictions of $0 < p,r < 1$ must be maintained. Enforcing these restrictions on (5) and (6) yields:

$$\max\left(1, \frac{\bar{x}}{1-\bar{x}}\right) < L_1 \tag{9}$$

This can be seen visually in the top plot of Figure 2.

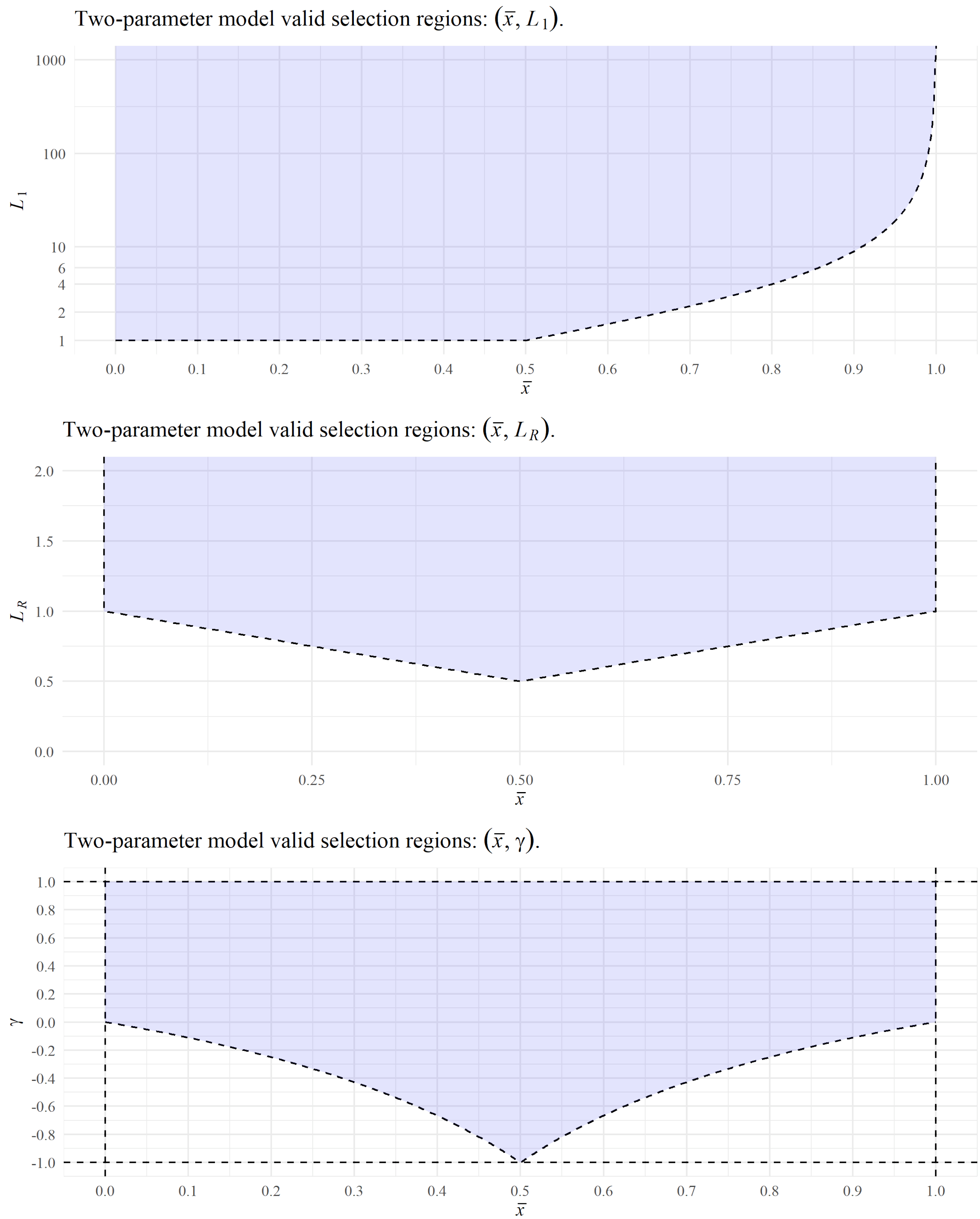


Figure 2: Valid regions for variable pairs in the two-parameter model. The top figure shows the valid regions when $(\bar{x}, L_1)$ are used to control the model, the middle figure when $(\bar{x}, L_R)$ are used, and the bottom figure when $(\bar{x}, \gamma)$ are used.

2.2 Error Rate and Relative Expected Burst Length

Table 1 also introduces the parameter, L_R, or relative expected burst length (REBL). REBL values are expected burst length values normalized by the expected burst length for the independent errors case, calculated while holding the error rate constant. In particular, given a fixed error rate, $\bar{x}$, the independent error model is characterized by $p^* = \bar{x}$ and $r^* = 1 - p^*$. This yields independent errors because it renders the two states as functionally identical, i.e., $P(X_n = 1|Z_n = G) = P(X_n = 1|Z_n = B) = p^* = \bar{x}$. The expected burst length in this independent errors case follows from (6):

$$L_1^* = \frac{1}{r^*} = \frac{1}{1-p^*} = \frac{1}{1-\bar{x}}. \tag{10}$$

Then the relative expected burst length L_R is defined as

$$L_R = \frac{L_1}{L_1^*}, \tag{11}$$

which can also be written as

$$L_R = \frac{1-\bar{x}}{r}. \tag{12}$$

It is then reasonably easy to see that

$$r(\bar{x}, L_R) = \frac{1-\bar{x}}{L_R} \tag{13}$$

and

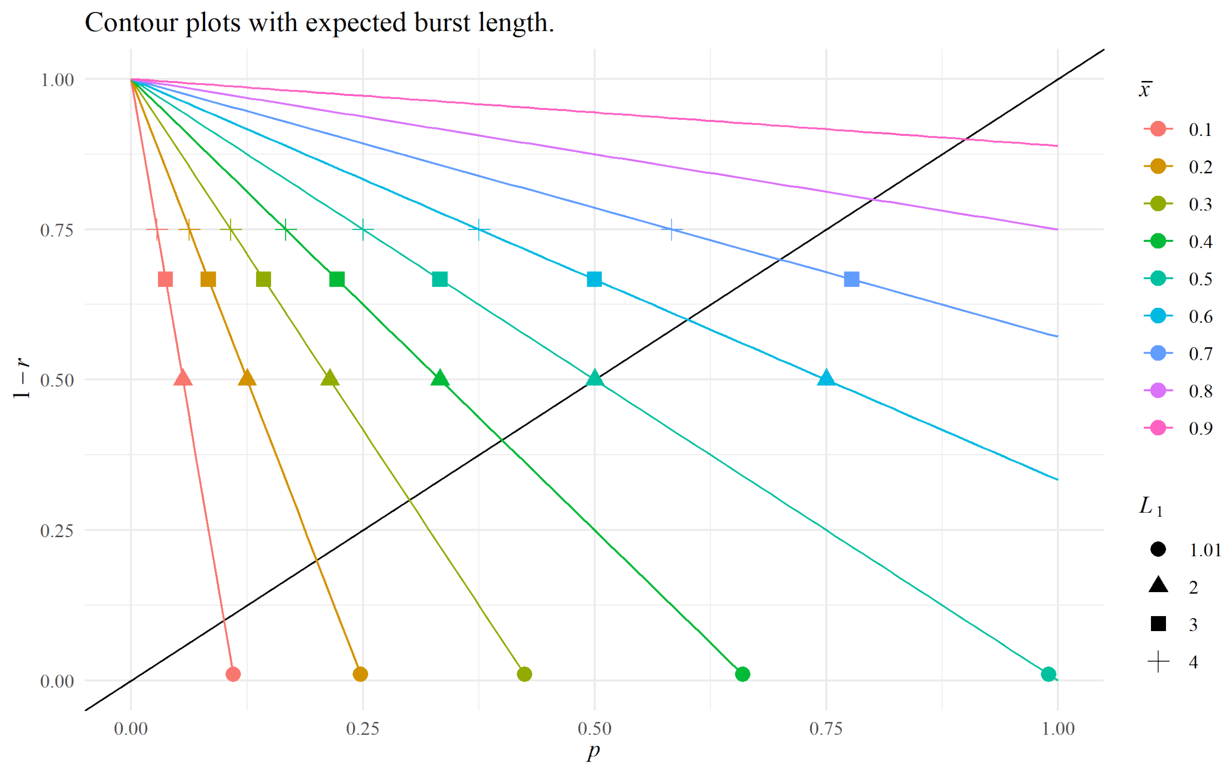
$$p(\bar{x}, L_R) = \frac{\bar{x}}{L_R}. \tag{14}$$

In the case of independent errors, $p = 1 - r$ and applying this constraint to (13) and (14) yields $L_R = 1$, as expected. Results (13) and (14) also lead to the the restrictions $0 < L_R < \infty$ and

$$\max(0, 1 - L_R) < \bar{x} < \min(1, L_R). \tag{15}$$

The justification for constraining $L_R < \infty$ rather than by a finite number follows the same reasoning as that given in Appendix B.

This restriction on $\bar{x}$ can be seen in the middle plot in Figure 2. In addition, Figure 3 shows how L_1 and L_R values are arranged in the plane that is defined by the transition probabilities p and $1 - r$. Compared to the L_1 values, the L_R values allow for meaningful control of burstiness across a wider range of error rates.



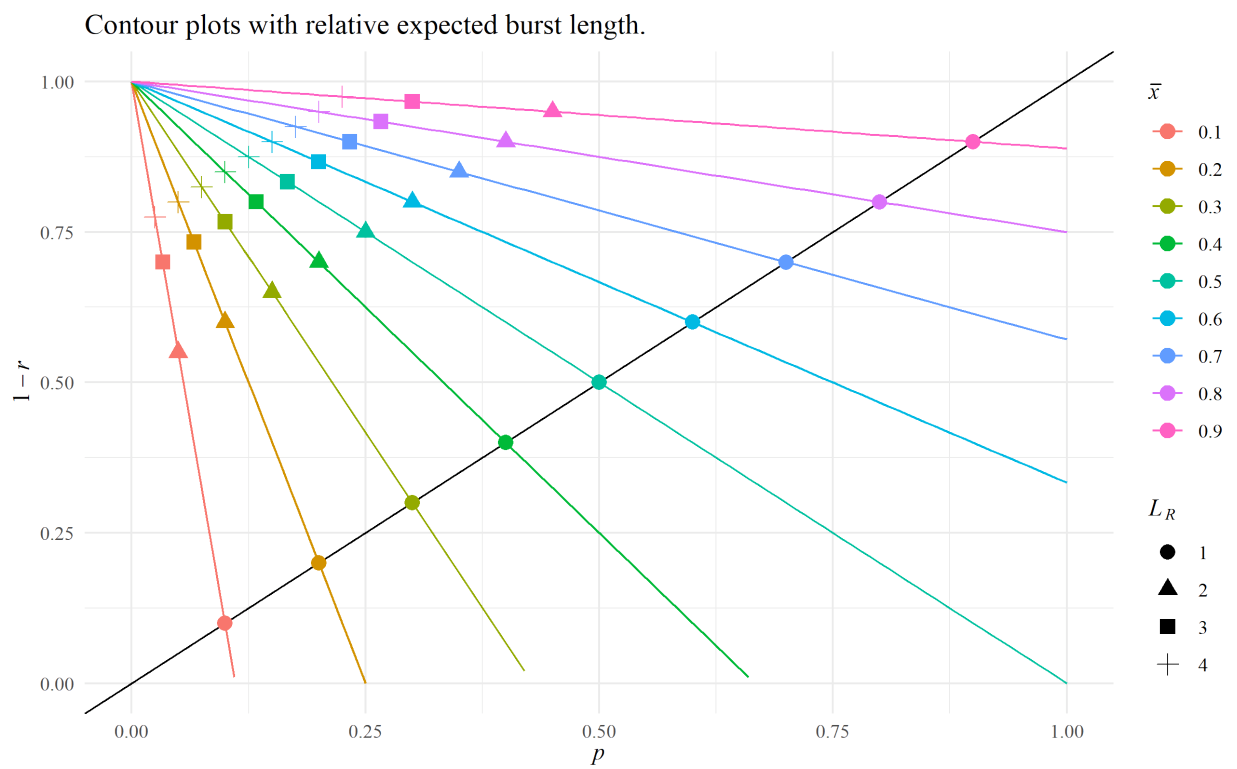


Figure 3: Contour plots showing lines of constant $\bar{x}$ on the p, $1-r$ plane. The top plot shows expected burst length (L_1) values where possible. Note that (9) shows why no L_1 levels are plotted for $\bar{x} = 0.8, 0.9$. In those cases, L_1 must be greater than 4 and 9 respectively. The bottom plot shows relative expected burst length (L_R) values. For L_R, at each given $\bar{x}$, the expected burst length is normalized by the expected burst length for the independent errors case ($p = 1-r$, shown with black line).

2.3 Error Rate and Lag-One Correlation

The lag-one correlation of a random process tells how its value at the current time step is related to its value at the previous time step. When the covariance is normalized by the variance of the process, the resulting correlation value ranges from -1 to 1 and forms a convenient measure of burstiness. The lag-one correlation γ is

$$\begin{aligned}\gamma =& \frac{\mathrm{Cov}\,(X_n, X_{n-1})}{\mathrm{Var}(X_n)} = \frac{\mathrm{E}(X_n - \bar{x})(X_{n-1} - \bar{x})}{\mathrm{E}(X_n - \bar{x})^2} = \frac{\mathrm{E}(X_n X_{n-1}) - \bar{x}^2}{\mathrm{E}(X_n^2) - \bar{x}^2} \\ =& \frac{(1-r)\bar{x} - \bar{x}^2}{\bar{x} - \bar{x}^2} = (1-r) - p.\end{aligned} \tag{16}$$

For independent errors, $p = 1 - r$ and (16) shows that this gives $\gamma = 0$, as expected. Note also that $0 < p, r < 1$ forces $-1 < \gamma < 1$.

Table 1 includes error rate ($\bar{x}$) and lag-one correlation (γ) as a useful pair of variables for model control so we consider this next. For the two-parameter model we have

$$\gamma(p, r) = (1 - r) - p, \tag{17}$$

which results in

$$p(\bar{x}, \gamma) = \bar{x} \cdot (1 - \gamma), \tag{18}$$

and

$$r(\bar{x}, \gamma) = 1 - \gamma - \bar{x} \cdot (1 - \gamma), \tag{19}$$

with the restriction of

$$\max\left(\frac{\bar{x}}{\bar{x} - 1}, \frac{\bar{x} - 1}{\bar{x}}\right) < \gamma < 1. \tag{20}$$

This relationship can be seen in the bottom plot of Figure 2.

3 THREE-PARAMETER MODEL

In the three-parameter model k remains fixed at $k = 1$ but $0 < h < 1$ is a free parameter. In other words, there are still no errors in the good state, but now the probability of an error occurring when in the bad state is $1 - h$. Table 2 shows some sets of parameters that can be used to control the model. In particular, the error rate, the expected burst length, and the proportion of time spent in the bad state are both meaningful and useful for controlling the model. Evaluating (3), (4), and (2) at $k = 1$ gives

Table 2: Sets of model parameters or error statistics for use with the three-parameter model.

Parameters	**Interpretation**	**Equations that convert to (p, r, h).**
$(p,\ r,\ h)$	Model state transition probabilities and probability of no error in the bad state	—
$(\bar{x},\ L_1,\ \pi_B)$	Error rate, expected burst length, proportion of time spent in bad state.	(26), (27), (28)
$(\bar{x},\ L_1,\ h)$	Error rate, expected burst length, probability of no error in the bad state.	(33), (34)

$$\bar{x}(p,r,h) = \frac{(1-h)p}{p+r}, \tag{21}$$

$$L_1(r,h) = \frac{1}{1-(1-r)(1-h)}, \tag{22}$$

$$\pi_B(p,r) = \frac{p}{p+r}. \tag{23}$$

We used the error correlation statistic, γ, with the two-parameter model but we do not promote its use with the three-parameter model. This statistic is less intuitive than error rate and expected burst length and is fully defined by those two parameters. In particular

$$\gamma(p,r,h) = \frac{r(1-h)(1-p-r)}{r+ph}. \tag{24}$$

This can be shown to also be

$$\gamma(\bar{x},L_1) = \frac{1-\frac{1}{L_1}-\bar{x}}{1-\bar{x}}. \tag{25}$$

Thus we focus only on the variable sets shown in Table 2.

3.1 Error Rate, Expected Burst Length, and Proportion of Time Spent in the Bad State

We will first consider using the error rate, the expected burst length, and the proportion of time spent in the bad state as controls for the model. Required model parameters are then given by:

$$h(\bar{x},\pi_B) = 1 - \frac{\bar{x}}{\pi_B}, \tag{26}$$

$$r(\bar{x}, L_1, \pi_B) = \frac{\bar{x}L_1 - \pi_B(L_1 - 1)}{\bar{x}L_1}, \tag{27}$$

and

$$p(\bar{x}, L_1, \pi_B) = \frac{\pi_B}{1 - \pi_B}\left(\frac{\bar{x}L_1 - \pi_B(L_1 - 1)}{\bar{x}L_1}\right). \tag{28}$$

Further, the restrictions that $0 < h, p < 1$ yield the following restrictions on $\bar{x}$ based on π_B and L_1:

$$0 < \bar{x} < \pi_B, \tag{29}$$

$$\frac{\pi_B(L_1 - 1)}{L_1} < \bar{x} < \left(\frac{L_1 - 1}{L_1}\right)\left(\frac{\pi_B^2}{2\pi_B - 1}\right). \tag{30}$$

Expressed together this means

$$\frac{\pi_B(L_1 - 1)}{L_1} < \bar{x} < \min\left(\pi_B, \frac{\pi_B^2(L_1 - 1)}{L_1(2\pi_B - 1)}\right). \tag{31}$$

This relationship is shown in the top plot in Figure 4. Note that

$$\pi_B < \frac{\pi_B^2(L_1 - 1)}{L_1(2\pi_B - 1)} \tag{32}$$

occurs when $\pi_B < \frac{L_1}{L_1+1}$.

3.2 Error Rate, Expected Burst Length, Probability of No Error in the Bad State

We now consider using the error rate, the expected burst length, and the probability of no error occurring when in the bad state (h) as controls for the model. Since we are using h as a control directly, we will only need two new relationships:

$$r(L_1, h) = \frac{1 - L_1 h}{L_1(1 - h)}, \tag{33}$$

$$p(\bar{x}, L_1, h) = \frac{\bar{x}(1 - L_1 h)}{L_1(1 - h)(1 - h - \bar{x})}. \tag{34}$$

By enforcing $0 < p, r < 1$ we find

$$1 < L_1 < \frac{1}{h}, \tag{35}$$

$$\bar{x} < \frac{L_1(1 - h)^2}{1 - L_1(2h - 1)}. \tag{36}$$

These relationships are shown in the bottom plot in Figure 4.

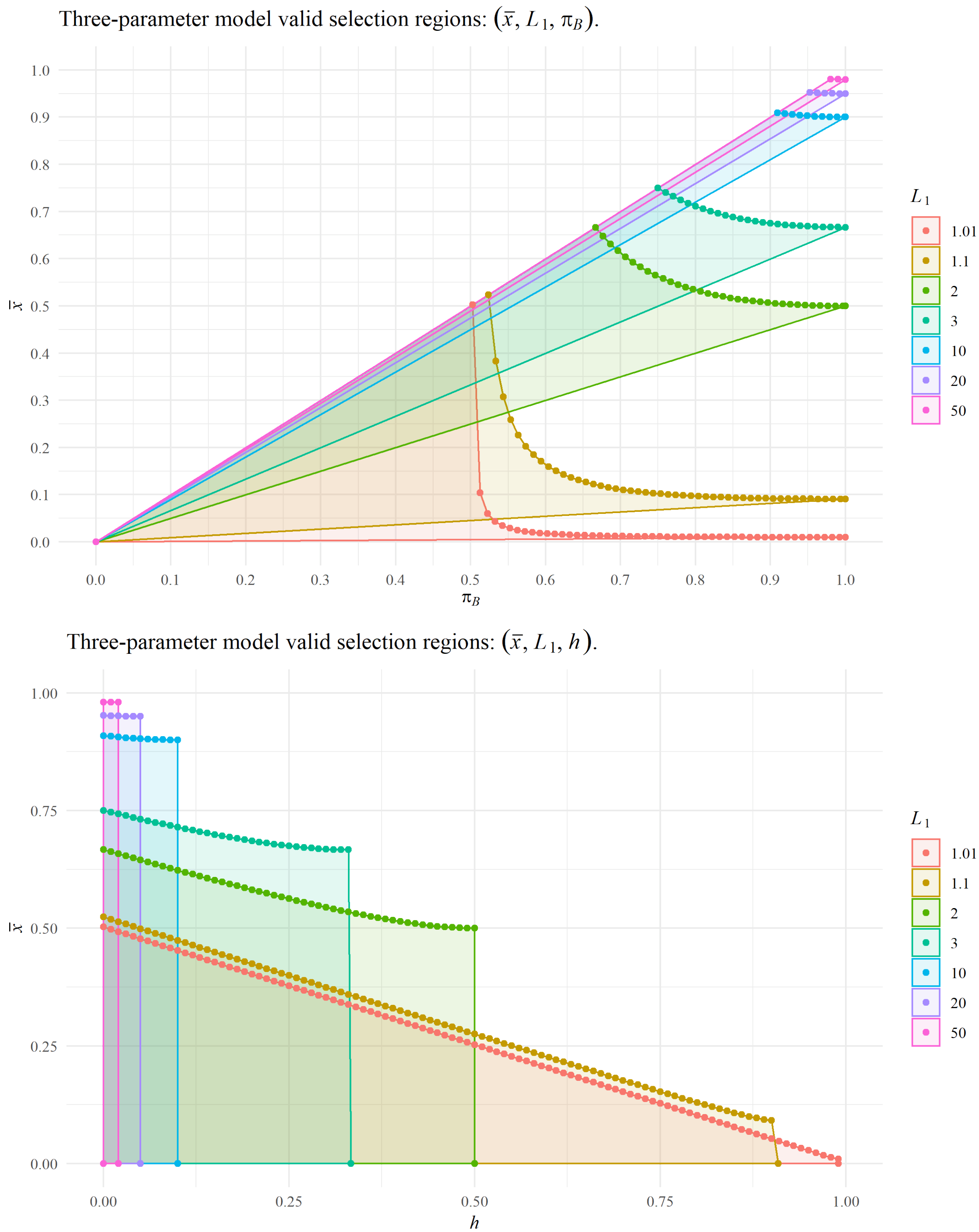


Figure 4: Valid regions for variable triples in the three-parameter model. The top plot shows valid regions when $(\bar{x}, L_1, \pi_B)$ are used while the bottom plot shows valid regions for $(\bar{x}, L_1, h)$. Note that only a small selection of valid L_1 values are displayed here.

4 FOUR-PARAMETER MODEL

We now return to the full four-parameter model, where $0 < h < k < 1$ so errors can occur in both the good and bad states. Table 3 shows sets of variables that might be used to control the model. It is worth noting that the model parameters of h or k appear in each of these sets of variables. Since $1-h$ and $1-k$ are conditional error rates they are reasonably interpretable. It is not straightforward to exclude h and k and still have four reasonably interpretable parameters.

Table 3: Sets of model parameters or error statistics for use with the four-parameter model.

Parameters	**Interpretation**	**Equations that convert to** (p,r,k,h)
$(p,\ r,\ h,\ k)$	State transition probabilities and conditional probabilities of error	—
$(\bar{x},\ L_1,\ \pi_B,\ h)$	Error rate, expected burst length, proportion of time in bad state, probability of no error in the bad state	(37), (40), (41)
$(\bar{x},\ L_1,\ \pi_B,\ k)$	Error rate, expected burst length, proportion of time in bad state, probability of no error in the good state	(42), (43), (44)

As one might expect, with four degrees of freedom the expressions for model parameters get fairly complicated. The valid regions for the three-parameter model were already fairly complex. This is further exhibited in the four-parameter case and as such valid regions are not given nor plotted as they are very difficult to present and interpret. Instead we suggest using the software referenced in Section 5 to determine valid ranges for variable values.

4.1 Error Rate, Expected Burst Length, Proportion of Time in Bad State, Probability of No Error in Bad State

We will first focus on using error rate, expected burst length, the proportion of time in the bad state, and the probability of no error when in the bad state to control the model. This leads to the following:

$$k(\bar{x},\pi_B,h) = \frac{1-\bar{x}-h\pi_B}{1-\pi_B}. \tag{37}$$

It is worth noting that due to the definition of π_B the following is true:

$$r = \frac{p(1-\pi_B)}{\pi_B}. \tag{38}$$

By rearranging the expression for L_1 in (4) one can find that

$$p = \frac{\bar{x}-L_1(h(1-h)\pi_B+k(1-k)(1-\pi_B))}{L_1(1-\pi_B)(h-k)^2}. \tag{39}$$

Substituting (37) into the above results in

$$p(\bar{x}, L_1, \pi_B, h) = \frac{L_1\left(\bar{x} - \pi_B\left(1-h\right)\right)\left(h\pi_B + \bar{x} - 1\right) + \left(1 - \pi_B\right)\left(\bar{x} - L_1 h \pi_B\left(1-h\right)\right)}{L_1\left(h + \bar{x} - 1\right)^2}. \tag{40}$$

With (38) we get

$$r(\bar{x}, L_1, \pi_B, h) = \frac{L_1(1-\pi_B)\left(\bar{x} - \pi_B\left(1-h\right)\right)\left(h\pi_B + \bar{x} - 1\right) + \left(1 - \pi_B\right)^2\left(\bar{x} - L_1 h \pi_B\left(1-h\right)\right)}{L_1 \pi_B\left(h + \bar{x} - 1\right)^2}. \tag{41}$$

4.2 Error Rate, Expected Burst Length, Proportion of Time Spent in Bad State, Probability of No Error in Good State

Similarly, we now focus on using error rate, expected burst length, the proportion of time in the bad state, and the probability of no error when in the good state to control the model. First we find

$$h(\bar{x}, \pi_B, k) = \frac{1 - \bar{x} - k(1-\pi_B)}{\pi_B}. \tag{42}$$

Using (39) and (42) we can get

$$r(\bar{x}, L_1, \pi_B, k) = 1 + \frac{\pi_B(\bar{x} - L_1(k^2 + k(2\bar{x} - 2) - \bar{x} + 1))}{L_1(1 - \bar{x} - k)^2} \tag{43}$$

and then

$$p(\bar{x}, L_1, \pi_B, k) = \left(\frac{\pi_B}{1 - \pi_B}\right)\left(1 + \frac{\pi_B(\bar{x} - L_1(k^2 + k(2\bar{x} - 2) - \bar{x} + 1))}{L_1(1 - \bar{x} - k)^2}\right). \tag{44}$$

4.3 Replacing the Proportion of Time Spent in the Bad State with the Good State

For either of the above sets of parameters it is trivial to use the proportion of time spent in the good state rather than the proportion of time spent in the bad state. One needs only to notice that $\pi_B = 1 - \pi_G$ and substitute that into any relevant equations.

5 SOFTWARE

We have written software to allow for easy use of the Gilbert-Elliot model. The software accepts sets of target error statistics, checks that they can produce valid model control parameter values (p,r,k,h), and then creates the corresponding output $\{X_n\}_{n=0}^{N-1}$ for the specified value of N. This output can then be used to impose bit errors or packet losses as desired.

The software can also accept a measured error pattern $\{\tilde{X}_n\}_{n=0}^{N-1}$, select the most appropriate model (two-,three-, or four-parameters) and then estimate the model parameters. This software can be found at `https://doi.org/10.5281/zenodo.7438482`.

As discussed in Section 4, the valid regions for controls in the four-parameter case are difficult to express and visualize. Even in the three-parameter case, the plots only show a limited selection of discrete values of L_1, and the expressions for the valid regions are not immediately intuitive. The software provides a tool that allows users to fix a selection of controls and determine a valid region for one free parameter. Some examples of this are included in Appendix A.

The software is primarily intended to be imported into other projects that involve burst errors. However we also developed a simple command line interface with a limited set of functionality. It allows users to determine valid regions for controls, translate model controls into model parameters, and also run simulations to generate error patterns of arbitrary length according to any set of controls presented in this memorandum. In Appendix A we provide some examples of how to use the command line interface.

6 CONCLUSION

We have presented and discussed two-, three-, and four-parameter Gilbert-Elliot models for bursty errors. We have drawn relationships between the model parameters (which are probabilities) and various meaningful statistics of the resulting error patterns. These relationships allow one to set model parameters so that the resulting error pattern has the desired error statistics. We also introduced software with simple tools to use the models in a variety of ways such as estimating model and parameters of an input error pattern and generating error patterns using any of the controls discussed in this paper.

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Appendix A: Software Examples

The following examples all assume the software has been downloaded from `https://doi.org/10.5281/zenodo.7438482` and installed per the included instructions.

In this example we would like to fix π_B and L_1 and determine the valid region for a selection of $\bar{x}$ in the three-parameter model. This is accomplished by running

```
gilbert-elliot --pi_B 0.7 --L1 3
--free-parameter xbar
```

which yields

```
xbar must be in:
Interval.Lopen(0.466666666666667, 0.700000000000000)
```

In this example we fix π_B, L_1, $\bar{x}$, and look for a valid value for h in the four-parameter model.

```
gilbert-elliot --xbar 0.6 --pi_B 0.7 --L1 3
--free-parameter h
```

This provides

```
h must be in:
Union(Interval.Ropen(0.142857142857143, 0.269069265858405),
Interval.open(0.530930734141595, 0.571428571428572))
```

Building on the last example we select a value of h from the valid region above and translate these model controls into the standard set of model parameters.

```
gilbert-elliot --xbar 0.6 --pi_B 0.7 --L1 3 --h 0.55
```

This results in

```
Model Parameters: {
        'p': 0.166666666666667, 'r': 0.0714285714285714,
        'h': 0.550000000000000, 'k': 0.0500000000000000
}
```

Finally we use the parameters above to generate a length 10000 error signal and save it to *myerrors.csv*.

```
gilbert_elliot.py --xbar 0.6 --pi_B 0.7 --L1 3 --h 0.55
--simulate --n_observations 10000 --output myerrors.csv
```

Appendix B: Expected Burst Length Derivations

We define $L_1 = E[L]$ where L is a geometric random variable with parameter $P(X_n = 0|X_{n-1} = 1)$. Then

$$L_1 = \frac{1}{P(X_n = 0|X_{n-1} = 1)} \ . \tag{45}$$

So we consider

$$\begin{aligned}
&P(X_n = 0|X_{n-1} = 1) = \frac{P(X_n = 0, X_{n-1} = 1)}{P(X_{n-1} = 1)} \\
&= \frac{P(X_n = 0, X_{n-1} = 1|Z_{n-1} = G)P(Z_{n-1} = G) + P(X_n = 0, X_{n-1} = 1|Z_{n-1} = B)P(Z_{n-1} = B)}{P(X_{n-1} = 1)} \\
&= \frac{P(X_n = 0|Z_{n-1} = G)P(X_{n-1} = 1|Z_{n-1} = G)\pi_G + P(X_n = 0|Z_{n-1} = B)P(X_{n-1} = 1|Z_{n-1} = B)\pi_B}{\bar{x}} \\
&= \frac{((1-p)k + ph)(1-k)r + ((1-r)h + rk)(1-h)p}{p+r} \cdot \frac{p+r}{(1-k)r + (1-h)p} \\
&= \frac{((1-p)k + ph)(1-k)r + ((1-r)h + rk)(1-h)p}{(1-k)r + (1-h)p}
\end{aligned} \tag{46}$$

So

$$L_1 = \frac{(1-k)r + (1-h)p}{(1-k)r((1-p)k + ph) + (1-h)p((1-r)h + rk)} \tag{47}$$

Note that the above assumes that sequence $\{X_n\}$ is infinite, while in reality all sequences would be of a finite length. This is done to simplify the logic and is a very reasonable approximation. If we treated each error burst with a finite maximum length, those maxima would then have to depend on the location of the burst within the finite-length sequence, which would make characterizing the expected burst length very challenging. In practice, assuming infinite length will not matter as the length of sequences of bits and packets is very large, especially when compared to error burst lengths in any channel that is successfully sending any information.

To make this explicit, consider the following. Let $v = P(X_n = 0|X_{n-1} = 1)$ and define $L_{1,m}$ as the expected value of L in the case where L can be no larger than m (i.e., in terms of a geometric distribution we stop either after the first success or after m trials). Then

$$L_{1,m} = \frac{1 - (1-v)^m}{v} \tag{48}$$

and

$$L_1 = \frac{1}{v} \tag{49}$$

so $L_{1,m}$ is always less than L_1. If we fix $\varepsilon > 0$ and consider $L_1 - L_{1,m} < \varepsilon$ we can find a value of m that satisfies this relation for any values of ε and v. In other words, if we define some equivalence tolerance, ε, and have some probability of exiting an error burst $v = P(X_n = 0|X_{n-1} = 1)$, we can find the value of m for which the finite expected value, $L_{1,m}$ is within ε of the infinite expected value L_1. Setting $\varepsilon = 1$ is reasonable as burst lengths are integers, though in practice a difference of a single bit or packet in a burst is likely negligible. Then we can see that if

$$m \geq \left\lceil \frac{\log(\varepsilon v)}{\log(1-v)} \right\rceil \tag{50}$$

then $L_1 - L_{1,m} < \varepsilon$ will be true.

It is easy to see that as v approaches 0, the constraint on m becomes larger. This is logical: as it becomes less likely that an error burst will end naturally, m must become larger so that the burst is not prematurely ended. In other words, m must become larger so that $L_{1,m}$ will behave similarly to L_1. Again, if error bursts are so long that this limit is hit it is extremely unlikely that anything would be successfully received through the communication system.

As an example consider the case where $v = P(X_n = 0|X_{n-1} = 1) = 0.01$ and $\varepsilon = 1$. Here $L_1 = 100$. If $m \geq 459$ then $L_1 - L_{1,m} < 1$ will be satisfied. In typical communications systems 1000 bits or packets is a reasonable bare minimum to consider, so the requirement $m \geq 459$ is a non-issue, and using L_1 in place of $L_{1,m}$ is an extremely reasonable approximation.

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15. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

The Gilbert-Elliot model is a popular and effective tool for modeling bursty (non-independent) errors in communication links. This memorandum provides linkages between model parameters and error statistics. The motivation is that these linkages can allow users to control models in order to obtain desired error statistics without any detailed understanding of Markov chains or probability. Features such as error rate and expected burst length are intuitive and also directly measurable in an error stream. This makes them natural candidates for controlling models after they are converted to the necessary model parameters (probabilities). We consider three different versions of the Gilbert-Elliot model and we present results for each. We also describe software that can be used to convert between error statistics and model parameters, to generate error patterns from a variety of variables, and also to estimate model parameters from an input error stream. This software is available at https://doi.org/10.5281/zenodo.7438482.

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