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An Analysis of Reception Variability Due to Terrain Multipath on Microwave Common Carrier Links

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AN ANALYSIS OF RECEPTION VARIABILITY DUE TO TERRAIN MULTIPATH ON MICROWAVE COMMON CARRIER LINKS

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The error variability of received PSK (MSK), QAM and QPR modulations on digital microwave common carrier links resulting from transmitted power variations and the dispersive effects of terrain multipath using a two-ray model is analyzed. These results and an earlier prediction of long-term (annual) average symbol error rate (SYE) for the above modulations are used to predict median annual received symbol error rate. Since the average SYE was used earlier to represent median conditions, average and new median SYE's are compared.

Year-to-year variability of received power and BER of common carrier microwave transmissions caused by variation of annual refractivity gradient distributions on off-axis paths is also analyzed, and it is concluded that this variability is not significant vis-a-vis the annual variability discussed above.

Key Words: Digital modulation, error rate variability, microwave links, reception variability, refractivity gradient distributions, terrain multipath.

1. Introduction and Background

Digitally modulated common carrier microwave terrestrial communication links, usually employ one of the following modulation types:

a) multiple phase-shift keying (M-PSK, $M \geq 4$)

b) multiple quadrature amplitude modulation (M-QAM)

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c) multiple quadrature partial response (M-QPR) or

d) minimum phase-shift keying (MSK).

At ITS we have developed modeling (e.g., Dutton, 1991) that predicts the symbol error rate for these digital modulation types; however, this modeling has developed an error result based on long term averages of signal-to-noise ratio (SNR) and signal-to-interference ratio (SIR). While this result is useful for predicting the receivability (e.g., relative to a bit-error-rate, BER, of 10^{-3}) of a microwave link in the presence of a persistent, long-term terrain multipath component when terrain reflection is treated as co-channel interference, it gives no information on the time-variability of this multipath. Hence, it is not possible to infer that the results given in Dutton (1991) for SYE are median annual results, other than by arbitrary assumption.

In the analysis that follows, it will be necessary to make some assumptions as well. First, it will be assumed that the SYE value discussed above represents the long-term average value; whereas, in fact, it is only a value determined by the long-term averages of SNR and SIR -- not necessarily the average itself. This is somewhat unfortunate, but it is the only estimate we have of the average SYE at this time. Second, the entire variance analysis is based on terrain reflection coefficient variability. Thus, while we are now able to infer some of the intersymbol interference (ISI) effects caused by terrain multipath, we are still unable to predict the ISI effects caused by atmospheric multipath. However, we will be able to predict the median SYE caused by both multipath types which is a decided improvement over our earlier situation. This is because the median (50 percent level) multipath is not likely to be affected by atmospheric layering, which occurs less than 50 percent of the time in the U.S.A. in all but the rarest instances (Dougherty and Dutton, 1981, pg. 17, Figure 8). In other words, we assume that the median SYE predicted from terrain multipath is representative of the median SYE for both terrain and atmospheric multipath types.

2. Variability of Terrain Multipath

We shall consider "instantaneous" power to be the power averaged over one symbol interval, T. The variance, $Var(p_r)$ of the received instantaneous power, p_r is then

$$Var(p_r) = \langle (p_r - \overline{p}_r)^2 \rangle = \langle p_r^2 \rangle - (\overline{p}_r)^2 \tag{1}$$

where $\overline{p}_r$ is the long-term average received power. Proakis (1989, Pg. 276) represents the minimum power in a pulse amplitude modulated (PAM) pulse by ξ/T, and gives the long-term average power in a (transmitted) PAM pulse, when there are $\sqrt{M}$ PAM levels ($M = 4, 16, 64, \ldots 2^{2n}$, where n is an integer ≥ 1) as

$$\frac{M-1}{3} \quad \frac{\xi}{T} \quad ,$$

where ξ is the energy in a pulse of duration, T. Here M is the order of M-QAM (composed of two PAM pulses in quadrature). Hence, the long-term average transmitted power, p_x, of an M-QAM pulse will be

$$\overline{p}_x = \frac{2}{3} \ (M-1) \ \frac{\xi}{T} \ . \tag{2}$$

M-QAM and M-QPR are the modulation types that are of variable amplitude. The $\overline{p}_x$ of M-QPR is given by (27). M-PSK and MSK do not vary in amplitude; thus, in these two cases

$$\overline{p_x} = p_x \ . \tag{3}$$

After transmission, the signal will suffer a power change, a, in the transmission medium before reception. In the atmosphere this change can be expressed as

$$a = \frac{a' g_{Td} g_{Rd}}{\ell_b \ell_e} \tag{4}$$

where a' is the atmospheric attenuation, g_{Td} and g_{Rd} are the power gains of the transmitter and receiver antennas, respectively, of the direct path between transmitter and receiver, ℓ_b is basic transmission loss along the path, and ℓ_e is equipment (waveguide) loss.

If we now represent the unwanted (interference) signal power as U, we can consider it at any given time to be the sum

$$U = I + \Delta U, \tag{5}$$

where I is the long-term (annual) mean interference power, and ΔU is a fluctuating component about I. We shall assume that all the received power, p_r, can be expressed as

$$p_r = W + U \tag{6}$$

where W is the wanted received power, and where, clearly

$$W = ap_x. \tag{7}$$

It is straightforward, using frequency domain analysis and the propagation medium transfer function of two-ray multipath, to obtain the "instantaneous" unwanted received power, as

$$U = ap_x \ \left\{ b^2\rho^2 + 2b\rho \cos\left(\Delta\phi\right) \right\} , \tag{8}$$

where ρ is the modulus (which is frequency dependent) of the terrain reflection coefficient (Dougherty and Dutton, 1986), $\Delta\phi$ (also frequency dependent) is the phase delay between the direct and reflected rays, and

$$b = \sqrt{\frac{g_{T1} g_{R1}}{g_{Td} g_{Rd}}} \quad . \tag{9}$$

In (9), g_{T1}, and g_{R1} are the antenna power gains in the directions of the reflected ray path from the transmitter and to the receiver, respectively, in the same great circle plane as the direct path. It should be noted that $\Delta\phi$ in (8) is essentially determined by the path difference between the reflected and direct rays, and $\Delta\phi$, as in earlier work (Dutton, 1991), is assumed to be uniformly distributed over time.

Under these conditions, then, the average received power, p_r, is

$$I = \overline{U} = a\overline{p}_x \quad (b^2\rho^2) \quad , \tag{10}$$

because $<\cos(\Delta\phi)> = 0$, p_x and $\Delta\phi$ are assumed independent, and ρ is assumed invariant over a long period of time. Although ***a*** is not invariant over a long period of time, it's variation occurs only in the tails of an annual distribution. Because our concern is primarily between the 20 and 80 percentiles of the annual distribution, ***a*** has also been assumed essentially invariant. After some algebra, we can obtain the expected value of U^2, $<U^2>$, from (8) as

$$\langle U^2 \rangle = a^2 \left\langle p_x^2 \right\rangle (2b^2\rho^2 + b^4\rho^4) \quad , \tag{11}$$

noting that $<\cos^2(\Delta\phi)> = 1/2$ and $<\cos(\Delta\phi)> = 0$.

Clearly from (11) in order to evaluate $<U^2>$, and eventually Var (U), we must first obtain $<p_x^2>$. From (3), it is apparent that we need only to evaluate $<p_x^2>$ for M-QAM and M-QPR. Since the M-QPR analysis can be obtained by analogy to the M-QAM situation, we shall first analyze M-QAM and present M-QPR results afterwards. The result for $<p_x^2>$ for M-QAM involves some cumbersome algebraic manipulation, as is shown in Appendix A; thus, we shall only state the M-QAM result,

$$\langle p_x^2 \rangle = 2\left[\frac{(3M-7)(M-1)}{15} + \frac{(M-1)^2}{9}\right]\left(\frac{\xi}{T}\right)^2 , \tag{12}$$

here. Thus, substituting (12) in (11), we obtain for M-QAM,

$$\langle U^2 \rangle = 2a^2\left[\frac{(3M-7)(M-1)}{15} + \frac{(M-1)^2}{9}\right]\left(\frac{\xi}{T}\right)^2 (2b^2\rho^2 + b^4\rho^4) . \tag{13}$$

We can obtain $(I)^2$ from (10) and (2) as

$$(I)^2 = a^2\left[\frac{2}{3}(M-1)\right]^2\left(\frac{\xi}{T}\right)^2 (b^4\rho^4) . \tag{14}$$

After some algebraic simplification, we can use (13) and (14) in (1) to obtain

$$Var(U) = S^2\left[\frac{3(3M-7)}{5(M-1)} + 1\right] b^2\rho^2 + \frac{1}{2} S^2\left[\frac{3(3M-7)}{5(M-1)} - 1\right](b^4\rho^4) . \tag{15}$$

In (15) we have used S to represent the desired rms received signal power which is simply

$$S = ap_x, \tag{16}$$

or, using (2),

$$S = \frac{2}{3}(M-1)\ \frac{\xi}{T}a\quad . \tag{17}$$

Note that S can be evaluated relative to a minimum spacing, D, between signal states in the M-QAM constellation, of D=2 (units unspecified, see Appendix A, Figures A1 and A2). This allows us to use a relative S, S_{rel}, with

$$S_{rel} = \frac{M-1}{3}\quad . \tag{18}$$

3. Error Rate Variability

From earlier work (Dutton, 1991), we have derived the SYE, P_{eM}, based on long-term means for modeled two-ray multipath as

$$P_{eM} = \frac{1}{K}\sum_{\ell=1}^{K} erfc\left(\sqrt{\frac{S}{N}}\sin\frac{\pi}{M} - \sqrt{\left(\frac{I}{S}\right)\left(\frac{S}{N}\right)}\quad \cos\left[\frac{2\pi(\ell-1)}{K}\right]\right)$$

$$- \frac{1}{K}\sum_{\ell=1}^{K} erf\left(\sqrt{\left(\frac{I}{S}\right)\left(\frac{S}{N}\right)}\cos\left[\frac{2\pi(\ell-1)}{K}\right]\right), \tag{19a}$$

for M-PSK (M ≥ 4) or MSK (equivalent to 4-PSK),

$$P_{eM} = \frac{1}{K}\left(1-\frac{1}{\sqrt{M}}\right)\sum_{\ell=1}^{K}\left[erfc\ (a_\ell) + erfc\ (b_\ell)\right]$$

$$- \frac{1}{4}\left(1 - \frac{1}{\sqrt{M}}\right)^2 \left\{\frac{1}{K}\sum_{\ell=1}^{K}\left[erfc(a_\ell) + erfc(b_\ell)\right]\right\}^2 , \tag{19b}$$

for M-QAM, and

$$P_{eM} = \frac{1}{K}\left(1 - \frac{1}{L^2}\right)\sum_{\ell=1}^{K}\left[erfc(c_\ell) + erfc(d_\ell)\ \right]$$

$$-\frac{1}{4}\left(1 - \frac{1}{L^2}\right)^2 \frac{1}{K}\left\{\sum_{\ell=1}^{K}\left[erfc(c_\ell) + erfc(d_\ell)\right]\right\}^2 , \tag{19c}$$

for M-QPR.

In (19a), (19b), and (19c), a value of K = 20 is sufficient to assure convergence in the evaluations, N is the rms noise power, and

$$L = \frac{\sqrt{M} + 1}{2} . \tag{20}$$

Also in (19b),

$$a_\ell = \frac{D}{\sqrt{8N}} - \sqrt{\left(\frac{S}{N}\right)\left(\frac{I}{S}\right)} \cos\left[\frac{2\pi(\ell-1)}{K}\right] \tag{21a}$$

and

$$b_\ell = \frac{D}{\sqrt{8N}} + \sqrt{\left(\frac{S}{N}\right)\left(\frac{I}{S}\right)} \cos\left[\frac{2\pi(\ell-1)}{K}\right] . \tag{21b}$$

In (19c)

$$c_\ell = \frac{D}{4\sqrt{\frac{N_o}{\pi}}} - \frac{\sqrt{I}}{\sqrt{2\frac{N_o}{\pi}}} \cos\left[\frac{2\pi(\ell-1)}{K}\right] , \tag{22a}$$

and

$$d_\ell = \frac{D}{4\sqrt{\frac{N_o}{\pi}}} + \frac{\sqrt{I}}{\sqrt{2\frac{N_o}{\pi}}} \cos\left[\frac{2\pi(\ell-1)}{K}\right] , \tag{22b}$$

where

$$N_o = \left(\frac{N}{S}\right)\frac{2}{\pi}\left(\frac{L^2-1}{3}\right) \tag{22c}$$

is a <u>dimensionless</u> result because in (22a), (22b) and (22c) we can assume a dimensionless D=1 minimum separation between signal states in the M-QPR signal constellation (see Appendix B, Figure B2).

At this point we make the major assumption that the forms of the expressions (19a), (19b) and (19c) for SYE based on long-term averages of SNR and SIR are the same as those for the instantaneous (and also the long-term mean) values of SYE, P_{eM}. This simply means that U can be substituted in (19a), (19b), (19c), (21a), (21b), (22a) and (22b) for I, respectively. For example, (19a) becomes

$$P'_{eM} = \frac{1}{K} \sum_{\ell=1}^{K} erfc\left(\sqrt{\frac{S}{N}} \sin \frac{\pi}{M} - \sqrt{\left(\frac{U}{S}\right)\left(\frac{S}{N}\right)} \cos\left[\frac{2\pi(\ell-1)}{K}\right]\right)$$

$$- \frac{1}{k} \sum_{\ell=1}^{K} erf\left(\sqrt{\left(\frac{U}{S}\right)\left(\frac{S}{N}\right)} \cos\left[\frac{2\pi(\ell-1)}{K}\right]\right)$$

for M-PSK and MSK. Note that the noise power, N, is assumed unchanging. Although N is slightly variable in time, it is relatively constant when compared to p_x (for M-QAM and M-QPR) or the terrain multipath variability. In equations such as the above, S is not replaced by W because either the transmitted power is constant, as it is in the M-PSK and MSK cases, or P_{eM} is dependent on the minimum difference between signal states, not the state itself, as in the cases of M-QAM and M-QPR. This minimum difference is constant with respect to all possible transmitted power states. Therefore, if *a* in (7) is also a constant (see, however, Section 7.), W is not inherently a variable in the error analysis equations.

The variance of P'_{eM}, Var (P'_{eM}) can, therefore, be expressed as a function of U alone for purposes of time-variability analysis. Thus, if

$$P'_{eM} = f(U) = f(I + \Delta U), \tag{23}$$

we can expand P'_{eM} in a Taylor series such that

$$P'_{eM} = f(I) + \frac{\partial f}{\partial U} | \Delta U + \ldots | \ . \qquad (24)$$

Following Crow et al. (1960, Pg 69), we can, therefore, say

$$Var(P'_{eM}) \approx \left(\frac{\partial f}{\partial U}\right)_I^2 Var(U) \ . \qquad (25)$$

Since W does not vary in the M-PSK and MSK cases, it is relatively straightforward to show that

$$Var(U) = 2S^2b^2\rho^2, \qquad (26)$$

for these modulations.

For M-QPR, a different set of formulas result, which can be derived in analogy with the M-QAM formulations. For M-QPR we obtain

$$\overline{P}_x = \frac{(L^2-1)}{3} \ \frac{\xi}{T} \ , \qquad (27)$$

whence, from (10)

$$I = \overline{U} = a \ \frac{(L^2-1)}{3} \ \left(\frac{\xi}{T}\right) \ (b^2\rho^2) \ . \qquad (28)$$

Furthermore, after some considerable algebra (See Appendix B)

$$\langle p_x^2 \rangle = \frac{(L^2-1)}{3} \left(\frac{17L^2-23}{30}\right) \left(\frac{\xi}{T}\right)^2 , \tag{29}$$

whence, from (11)

$$\langle U^2 \rangle = a^2 \frac{(L^2-1)}{3} \left(\frac{17L^2-23}{30}\right) \left(\frac{\xi}{T}\right)^2 (2b^2\rho^2+b^4\rho^4) . \tag{30}$$

Using (1) and (16), we obtain

$$Var(U) = S^2 \left[\frac{7L^2-13}{10(L^2-1)}\right] (b^4\rho^4) + S^2 \left[\frac{17L^2-23}{10(L^2-1)}\right] (2b^2\rho^2) , \tag{31}$$

for M-QPR. It is important to note that while the M-QAM result (15) assumes the transmitted signal is transmitted with any of the M possible amplitudes being equally likely, the M-QPR result of (31) requires an assumption that the distribution of amplitudes on each partial response axis is triangular (Simpson) distributed (Proakis, 1989, Pg 544).

After some algebra in an expression for P'_{cM} analogous to (19a) which is, from (23), a function of ***U*** rather than ***I,*** we obtain

$$\frac{\partial P'_{eM}}{\partial U}\bigg|_I = \frac{\partial f}{\partial U}\bigg|_I = \frac{1}{K}\sum_{\ell=1}^{K}\frac{1}{2S}\sqrt{\left(\frac{S}{I}\right)\left(\frac{S}{N}\right)}\cos\left[\frac{2\pi(\ell-1)}{K}\right]\frac{2}{\sqrt{\pi}}e^{-y_\ell^2} \tag{32}$$

$$-\frac{1}{K}\sum_{\ell=1}^{K}\frac{1}{2S}\sqrt{\left(\frac{S}{I}\right)\left(\frac{S}{N}\right)}\cos\left[\frac{2\pi(\ell-1)}{K}\right]\frac{2}{\sqrt{\pi}}e^{-x_\ell^2}$$

for M-PSK or MSK. In (32)

$$y_\ell = \sqrt{\frac{S}{N}}\sin\frac{\pi}{M} - \sqrt{\left(\frac{I}{S}\right)\left(\frac{S}{N}\right)}\cos\left[\frac{2\pi(\ell-1)}{K}\right] , \tag{33a}$$

and

$$x_\ell = \sqrt{\left(\frac{I}{S}\right)\left(\frac{S}{N}\right)}\cos\left[\frac{2\pi(\ell-1)}{K}\right] . \tag{33b}$$

For M-QAM, we obtain

$$\frac{\partial f}{\partial U}\bigg|_I = \frac{1}{KS}\left(1-\frac{1}{\sqrt{M}}\right)\sum_{\ell=1}^{K}\frac{1}{\sqrt{\pi}}\sqrt{\left(\frac{S}{I}\right)\left(\frac{S}{N}\right)}\cos\left[\frac{2\pi(\ell-1)}{K}\right](e^{-a_\ell^2}-e^{-b_\ell^2})$$

(34)

$$-\frac{1}{2KS}\left(1-\frac{1}{\sqrt{M}}\right)^2\left(\frac{1}{K}\sum_{\ell=1}^{K}[erfc(a_\ell)+erfc(b_\ell)]\right)\left\{\sum_{\ell=1}^{K}\frac{1}{\sqrt{\pi}}\sqrt{\left(\frac{S}{I}\right)\left(\frac{S}{N}\right)}\cos\left[\frac{2\pi(\ell-1)}{K}\right](e^{-a_\ell^2}-e^{-b_\ell^2})\right\} ,$$

where a_ℓ and b_ℓ are given by (21a) and (21b), respectively.

For M-QPR, we obtain

$$\left.\frac{\partial f}{\partial U}\right|_{I} = \frac{1}{K}\left(1-\frac{1}{L^2}\right)^2 \sum_{\ell=1}^{K} \frac{1}{\sqrt{2IN_o}} \cos\left[\frac{2\pi(\ell-1)}{K}\right] (e^{-c_\ell^2}-e^{-d_\ell^2})$$

$$-\frac{1}{2K}\left(1-\frac{1}{L^2}\right)^2 \left(\frac{1}{K}\sum_{\ell=1}^{K}[erfc(c_\ell)+erfc(d_\ell)]\right)\left\{\sum_{\ell=1}^{K}\frac{1}{\sqrt{2IN_o}}\cos\left[\frac{2\pi(\ell-1)}{K}\right](e^{-c_\ell^2}-e^{-d_\ell^2})\right\}, \tag{35}$$

where c_ℓ and d_ℓ are given by (22a) and (22b), respectively.

Through the use of (25), we can obtain the time variability, Var (P'_{eM}), for the various modulation types. For M-PSK or MSK (the equivalent of M-PSK for M=4), we obtain, by using (26) and (32) in (25)

$$Var(P'_{eM}) = \frac{2b^2\rho^2}{\pi K^2}\left\{\sum_{\ell=1}^{K}\sqrt{\left(\frac{S}{I}\right)\left(\frac{S}{N}\right)}\cos\left[\frac{2\pi(\ell-1)}{K}\right](e^{-y_\ell^2}-e^{x_\ell^2})\right\}^2 . \tag{36}$$

For M-QAM, we obtain, by using (15) and (34),

$$Var(P'_{eM}) = \left(\frac{\partial f}{\partial U}\right)_I^2 Var(U) = \frac{1}{\pi K^2}\left(1-\frac{1}{\sqrt{M}}\right)^2 \left\{\left[\sum_{\ell=1}^{K}\sqrt{\left(\frac{S}{I}\right)\left(\frac{S}{N}\right)}\cos\left[\frac{2\pi(\ell-1)}{K}\right](e^{-a_\ell^2}-e^{-b_\ell^2})\right]\right.$$

$$\left.\cdot\left(1-\left(1-\frac{1}{\sqrt{M}}\right)\frac{1}{2K}\sum_{\ell=1}^{K}[erfc(a_\ell)+erfc(b_\ell)]\right)\right\}^2 \tag{37}$$

$$\cdot\left\{\left[\frac{3(3M-7)}{5(M-1)}+1\right]b^2\rho^2+\frac{1}{2}\left[\frac{3(3M-7)}{5(M-1)}-1\right](b^4\rho^4)\right\} ,$$

For M-QPR, we obtain, by using (31) and (35),

$$Var(P'_{eM}) = \left(\frac{\partial f}{\partial U}\right)^2_I Var(U) = \frac{1}{K^2}\left(1-\frac{1}{L^2}\right)^2 \left\{\left(\sum_{\ell=1}^{K} \frac{1}{\sqrt{2IN_o}} \cos\left[\frac{2\pi(\ell-1)}{K}\right] (e^{-c_\ell^2} - e^{-d_\ell^2})\right)\right.$$

$$\left. \cdot \left(1-\left(1-\frac{1}{L^2}\right)\frac{1}{2K}\sum_{\ell=1}^{K}[erfc(c_\ell) + erfc(d_\ell)]\right)\right\}^2 \tag{38}$$

$$\cdot \left\{(2S^2b^2\rho^2)\left[\frac{17L^2-23}{10(L^2-1)}\right] + (b^4\rho^4)\ S^2\left[\frac{7L^2-13}{10(L^2-1)}\right]\right\}.$$

4. Error Rate Cumulative Distribution Function

The distribution of the symbol error rates, P_{eM}, over time must inherently be unlike the distribution of received power or signal level (RSL) over time. This is because although received power itself is never negative, its logarithm can be; hence, the RSL in decibels is often regarded as a normal distribution (the power itself is lognormally distributed). Symbol error rates, however, are probabilities, and are therefore, constrained to be between zero and unity; hence, the standard normal or lognormal distributions cannot be directly applied to P_{eM}.

Following Beckmann (1967, pp. 415-16), a doubly-truncated (truncated at zero and unity) normal distribution can be used to describe the cdf of P_{eM}. In this case, a "truncating factor", *C*, must be obtained such that

$$C = \int_0^1 \frac{1}{\sqrt{2\pi Var(P'_{eM})}} \exp\left[-\frac{(P'_{eM}-P_{eM})^2}{2Var(P'_{eM})}\right] dP'_{eM} = 1\ , \tag{39}$$

where P_{eM}, given for various modulations by (19a), (19b), and 19c), is assumed (as discussed in Section 1.0) to be the long-term mean value of P'_{eM}. After evaluation of (39), we obtain

$$C = \frac{2}{erf\left[\frac{P_{eM}}{\sqrt{2Var(P'_{eM})}}\right] + erf\left[\frac{1-P_{eM}}{\sqrt{2Var(P'_{eM})}}\right]} \quad . \tag{40}$$

The probability that some random symbol error, SYE, exceeds P'_{eM} can be expressed as

$$Pr(SYE > P'_{eM}) = \frac{C}{\sqrt{2\pi Var(P'_{eM})}} \int_{P_{eM}}^{1} \exp\left[-\frac{(P'_{eM} - P_{eM})}{2Var(P'_{eM})}\right] dP'_{eM} \; . \tag{41}$$

Note that one can either use the percent of time, 100 Pr (SYE>P'_{eM}), that an error rate is exceeded, or the percent of time 100 [1 - Pr(SYE>P'_{eM})] that the error rate is not exceeded to define a given boundary for receivability. When one wishes to use median conditions (50% of the time), either percentile yields the same result. The symbol error rate, $P'_{eM\%}$, at any exceedance percentile can be obtained from (41) as

$$P'_{eM\%} = P_{eM} + \sqrt{2Var(P'_{eM})}\; erf^{-1}\left\{\frac{-2Pr(SYE > P'_{eM})}{C} + erf\left[\frac{1-P_{eM}}{\sqrt{2\,Var(P'_{eM})}}\right]\right\}, \tag{42}$$

or, for median conditions

$$P'_{eM,50} = P_{eM} + \sqrt{2\;Var(P'_{eM})}\; erf^{-1}\left\{-\frac{1}{C} + erf\left[\frac{1-P_{eM}}{\sqrt{2Var(P'_{eM})}}\right]\right\}. \tag{43}$$

5. Comparison of Results

If we compare $P_{eM,50}$, the median prediction, and P_{eM}, the average prediction of SYE for some reasonable multipath conditions, we obtain the results shown in Table 1.

Table 1

Predicted Average and Median Symbol Error Rates (SYE) for
$\rho = 0.1$ and $b = 1$
and Various Signal-to-Noise Ratios, SNR

SNR	8-PSK P_{eM}	SYE $P'_{eM,50}$	16-QAM P_{eM}	SYE $P'_{eM,50}$	64-QAM P_{eM}	SYE $P'_{eM,50}$
10	0.103	0.195	0.237	0.375	0.637	0.593
15	$7.02X10^{-3}$	$5.65X10^{-2}$	$3.99X10^{-2}$	0.280	0.423	0.495
20	$1.05X10^{-5}$	$2.84X10^{-4}$	$6.13X10^{-4}$	$1.50X10^{-2}$	0.199	0.495
30	$3.89X10^{-17}$	$5.15x10^{-17}$	$3.80x10^{23}$	$8.91x10^{-21}$	$2.93X10^{-3}$	0.206
40	$< 10^{-25}$	$< 10^{-25}$	$< 10^{-25}$	$< 10^{-25}$	$1.42X10^{-15}$	$8.71X10^{-13}$

Some tentative conclusions can be drawn from Table 1. It appears that for SYE $< 10^{-6}$, the average or median value could be used interchangeably since neither changes receivability results appreciably. However, for SYE$> 10^{-6}$ error probabilities are appreciably different. In the case of error probabilities, relative differences are not as consequential as actual differences, since most small errors (SYE$< 10^{-6}$) can be eliminated by equalization and/or diversity. Large actual differences cannot be so easily remedied and, furthermore, can change a prediction of BER from receivable to unreceivable. There are two such cases in Table 1; viz., at SNR = 20 dB for

16-QAM modulation, and, at SNR = 30 dB for 64-QAM modulation.[2] It is, of course, the SYE's near threshold (SYE$>10^{-6}$) are often of most concern in many applications because of the increasing possibility of loss of synchronization at higher error rates.

6. Refractivity Gradient Distribution Variability

The variability of the received signal power caused by inherent transmitted power variation and terrain multipath was discussed in Sections 2.0, 3.0 and 4.0. This variability was described as "long-term" variability, which could generally be expected to be an annual variation. It is important to recognized, however, that there is a year-to-year variability component as well, that is caused by the variation in the factor, *a*, of equation (4). In the preceding sections, *a* was treated as essentially a constant but it is necessary to determine whether such an assumption is warranted or not. We are not talking here of the small within-year variability of *a*, which is being caused primarily by rare events such as rain attenuation, and can be justifiably ignored. We are talking instead of the variability from year-to-year of the refractivity gradient distribution at a given percentile (generally at the 50% level) which could cause ray-path changes and changes in path clearance.

If there are no drastic changes in ray-path clearance (i.e., enough of a change to encounter an obstacle that was cleared before), RSL changes caused by year-to-year variability of the surface refractivity gradient distribution can be shown to be relatively negligible compared to the annual variation. If we denote the within year (annual) variation of received wanted power as W_{wy}, and of received unwanted power as U_{wy}, then,

---

[2] The SNR = 30 dB, 64-QAM value of $P_{eM} = 2.93 \times 10^{-3}$ translates to a dBER = 4.88×10^{-4} if one bit error per symbol error is assumed.

$$Var\ (W) \approx (\overline{p}_x)^2\ Var(a) + Var\ (W_{wy})\ , \tag{44a}$$

$$Var\ (U) \approx (\overline{p}_x)^2\ Var(a) + Var\ (U_{wy})\ , \tag{44b}$$

provided ***U*** is caused by nearby multipath components such as in the two-ray model. In (44a) and (44b), Var (W) and Var (U) would be the true (i.e., multiyear) long-term variability of W and U, respectively, and $\overline{p}_x$ is the long-term average transmitted power. Figures 1 and 2 show five years of annual distributions of the surface to 300 m refractivity gradient for Sterling, VA (essentially Washington, DC) and Oakland, CA, as examples of the many such annual distribution sets throughout the contiguous United States of America (CONUS) that we have prepared from data provided by the U.S. Weather Service. Figures 1 and 2 are typical in that, except for the tails of the distributions, there is very little year-to-year variability of the distributions. Therefore, in the region of the distributions near the mean (the region of interest to us), little variation in refractivity gradient from year to year means little variation in the basic transmission loss, ℓ_b, in (4). Hence, Var (a) will be small and we can reasonably assume that (44a) and (44b) can be rewritten as

$$Var\ (W) \approx Var\ (W_{wy})\ , \tag{45a}$$

and

$$Var\ (U) \approx Var\ (U_{wy})\ , \tag{45b}$$

respectively, implying that the contribution of year-to-year variability need not be considered in this case.

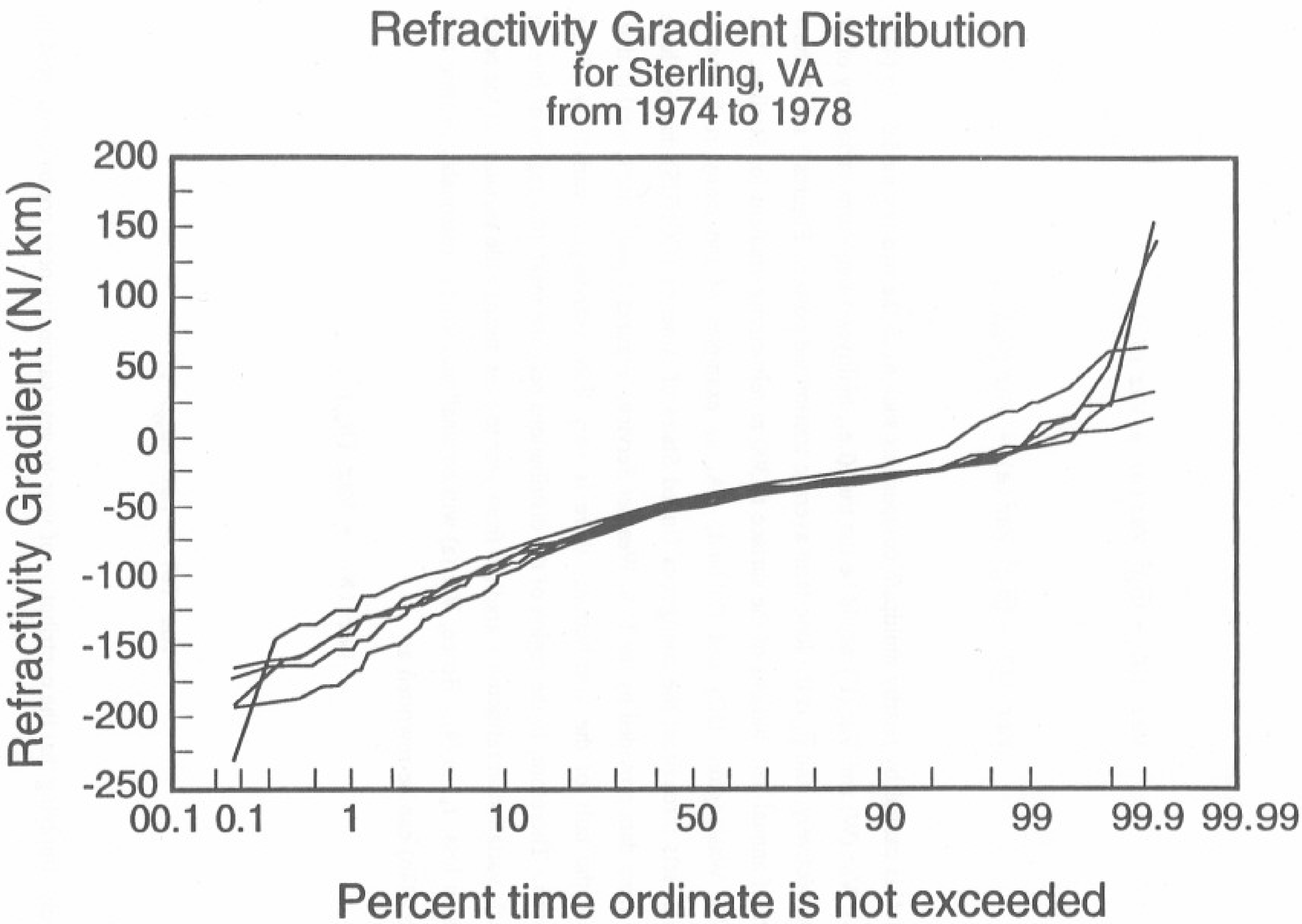

Figure 1 : Five years of annual refractivity gradient distributions (surface to 300m) for Sterling, VA.

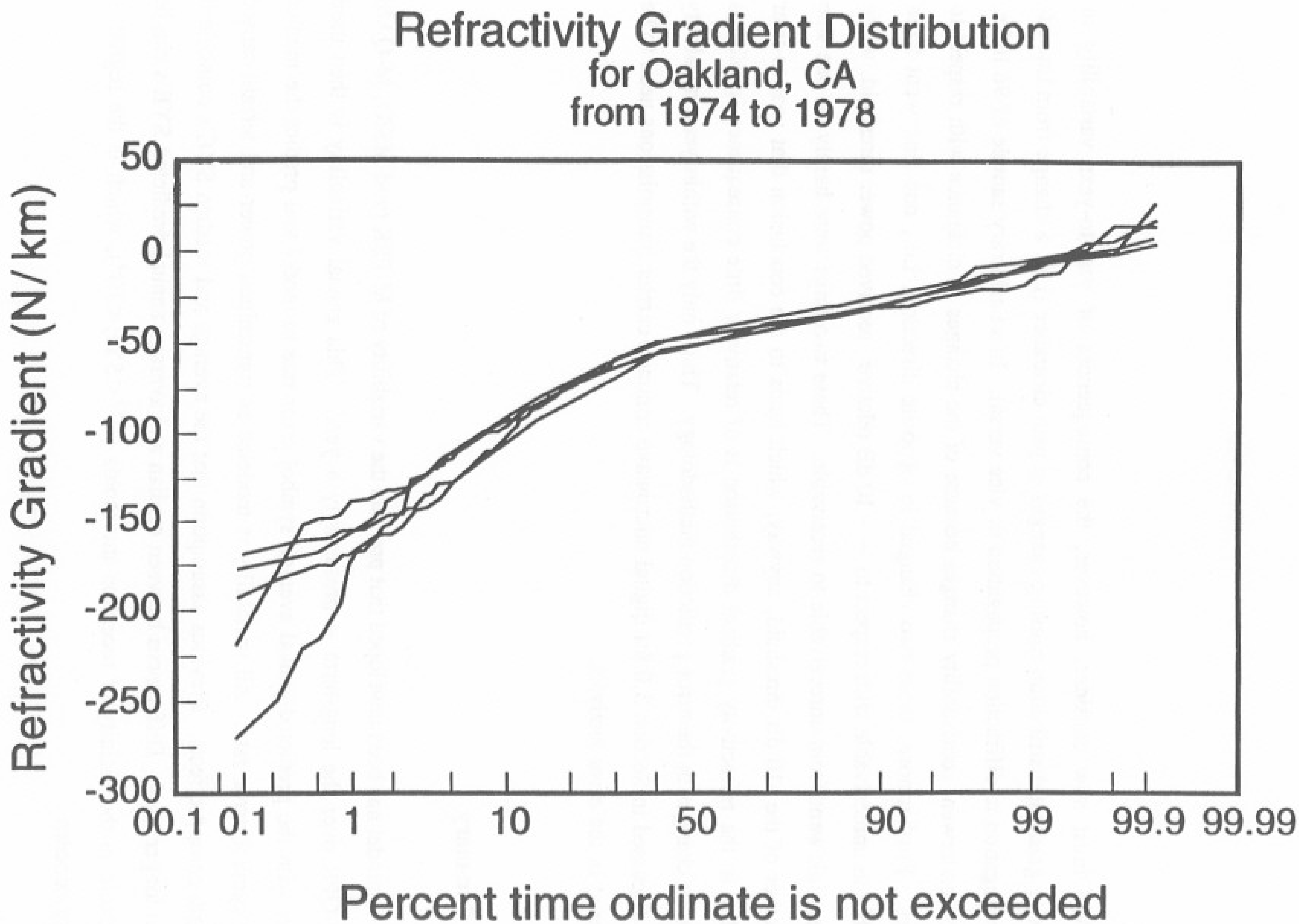

Figure 2: Five years of annual refractivity distributions (surface to 300m) for Oakland, CA.

We must now consider, however, the consequences of year-to-year variability in refractivity gradient distribution causing changes in path clearance (i.e., a change from line-of-sight propagation to diffraction propagation or vice versa). In an arbitrary sample of 96 links, only two underwent receivability changes because of the changes in clearance with respect to obstacles. Furthermore, those two changed in opposite direction, i.e., one link went from receivable to unreceivable with respect to ~ - 10 dB relative[3] received power threshold, while the other link went from unreceivable to receivable. These two links were barely on one side or the other of the -10 dB threshold, anyway, which leads to the conclusion that year-to-year variability in the refractivity gradient distribution is of relatively little consequence insofar as influencing changes in the error prediction methodology. Thus, only the within-year variability effects discussed in Section 3.0 for digital microwave common carrier transmissions need to be considered in the error analysis.

7. Summary

A model has been developed that predicts the variability of M-PSK (and MSK), M-QAM and M-QPR over the long-term -- presumably a year. This annual variability is then used together with the predicted annual average symbol error rate to model and predict the median annual symbol error rate. All variability is modeled as transmitted power and terrain-caused multipath caused effects. Previous assumption that the average and median SYE's coincide is now no longer used. Differences between median and average annual predicted SYE's can be appreciable in the vicinity of reception thresholds ($10^{-2} < SYE < 10^{-4}$), which is the region of primary concern.

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[3] Relative to the noise background.

Year-to-year variability of the refractivity gradient has been shown to be largely inconsequential relative to the microwave reception error predictions. This is essentially because the year-to-year variability of the refractivity gradient distributions is slight for most all of the distributions that are expected to be encountered nationwide and worldwide. Thus, propagation modeling need be revised only to include the within-year (annual) variability effects.

8. References

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APPENDIX A

Derivation of the mean-square transmitted power for M-QAM

Following Proakis (1989, pg. 272), an m-PAM pulse in the interval T, $0<t<T$, can be represented by

$$s_i(t) = A_i \, Re\left[u(t)\, e^{j2\pi f_c t}\right] , \quad i=1,2,\ldots,m \quad , \tag{A1}$$

where $s_i(t)$ is the signal waveform, A_i is the signal amplitude taking on the discrete levels

$$A_i = 2i - 1 - m \, . \tag{A2}$$

Again following Proakis (1989, Pg. 273), the "instantaneous" power (i.e., in the interval T) of such a waveform is given by

$$p_{xi} = \frac{1}{T}\int_o^T s_i^2(t)\, dt = \frac{A_i^2}{2}\,\frac{1}{T}\int |u(t)|^2 \, dt = A_i^2 \, \frac{\xi}{T} \quad , \tag{A3}$$

and the average power of the m-PAM transmission is

$$\overline{p}_{xi} = \left\langle A_i^2 \right\rangle \frac{\xi}{T} \quad , \tag{A4}$$

where ξ is the energy in a pulse of duration, T.

An M-QAM signal is composed of two m-PAM signals ($M=m^2$) in quadrature and added vectorially. As a result, the average power, p_x, of an M-QAM signal is given by

$$\overline{p_x} = \overline{p_{xi}} + \overline{p_{xj}} = \langle A_i^2 + A_j^2 \rangle \frac{\xi}{T} = 2 \langle A_i^2 \rangle \frac{\xi}{T} \quad , \tag{A5}$$

where p_{xi} is the average power on one PAM axis, and p_{xj} (j = 1, 2, . . . , *m*) is the average power on the quadrature axis. The result (2) in the main text can also be obtained from Proakis (1989, Pg. 276) using (A5). The instantaneous power, p_x, of an M-QAM transmission is also readily obtained as

$$p_x = (A_i^2 + A_j^2) \frac{\xi}{T} \quad . \tag{A6}$$

Thus, the mean-square transmitted power, $<p_x^2>$, is given by

$$\langle p_x^2 \rangle = \left(\frac{\xi}{T}\right)^2 \langle (A_i^2 + A_j^2)^2 \rangle \tag{A7}$$

$$= \left(\frac{\xi}{T}\right)^2 (\langle A_i^4 \rangle + 2 \langle A_i^2 A_j^2 \rangle + \langle A_j^4 \rangle) \quad .$$

Because the two quadrature axes are identical m-PAM modulations and, being orthogonal, are independent of each other, we can say

$$\langle p_x^2 \rangle = \left(\frac{\xi}{T}\right)^2 (\langle A_i^4\rangle + 2 \langle A_i^2 \rangle\langle A_j^2\rangle + \langle A_j^4 \rangle)$$

$$= 2 \left(\frac{\xi}{T}\right)^2 (\langle A_i^4 \rangle + \langle A_i^2 \rangle^2) \tag{A8}$$

The discrete level notation of (A2) means that, for a minimum separation of 2 "units" (or D=2 as in equations (21a) and (21b) of the main text) between states in the signal constellation (see Figures A1 and A2),

$$\langle A_i^4 \rangle = \langle (2i - 1 - m)^4 \rangle \ , \tag{A9}$$

which can alternatively be written as

$$\langle A_i^4 \rangle = \frac{2}{m} \sum_{i=1}^{m/2} (2i-1)^4 \quad . \tag{A10}$$

We can write

$$(2i - 1)^4 = 16i^4 - 32i^3 + 24i^2 - 8i + 1 \quad , \tag{A11}$$

and

$$\frac{2}{m} \sum_{i=1}^{m/2} (2i-1)^4 = \frac{32}{m} \sum_{i=1}^{m/2} i^4 - \frac{64}{m} \sum_{i=1}^{m/2} i^3 + \frac{48}{m} \sum_{i=1}^{m/2} i^2 - \frac{16}{m} \sum_{i=1}^{m/2} i + \frac{2}{m} \left(\frac{m}{2}\right) . \tag{A12}$$

The finite sums of integers to constant integer powers are generally well known, so that the substitution of these expressions into (A12), along with some tedious algebra, eventually results in

$$\langle A_i^4 \rangle = \frac{(3m^2-7)\ (m^2-1)}{15} \quad , \tag{A13}$$

or since $M = m^2$

$$\langle A_i^4 \rangle = \frac{(3M-7)\ (M-1)}{15} \quad . \tag{A14}$$

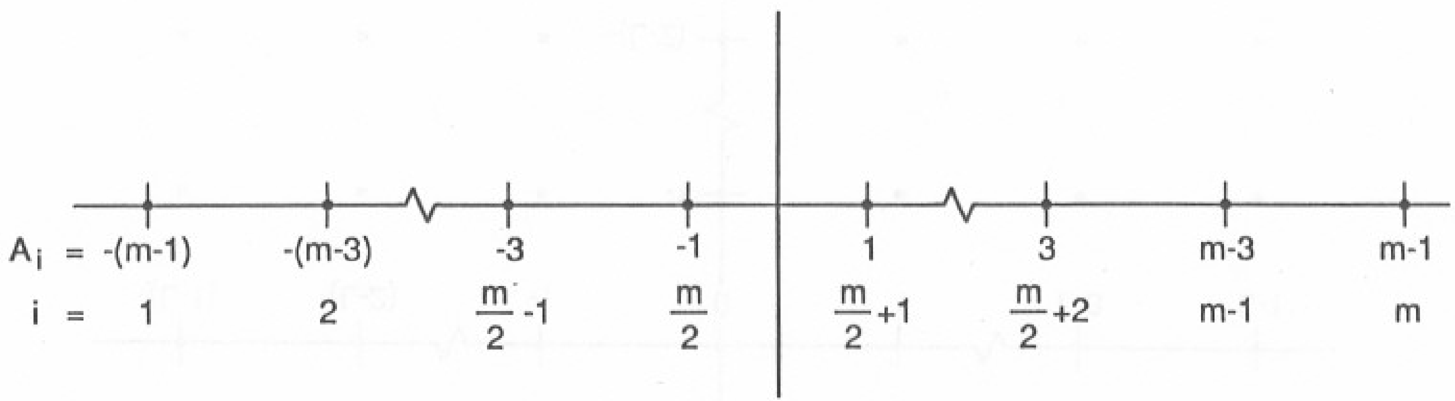

Figure A1. *Sketch of a PAM amplitude, A_i, axis for $i = 1$ to m levels, and an amplitude separation, $D = 2$ between levels.*

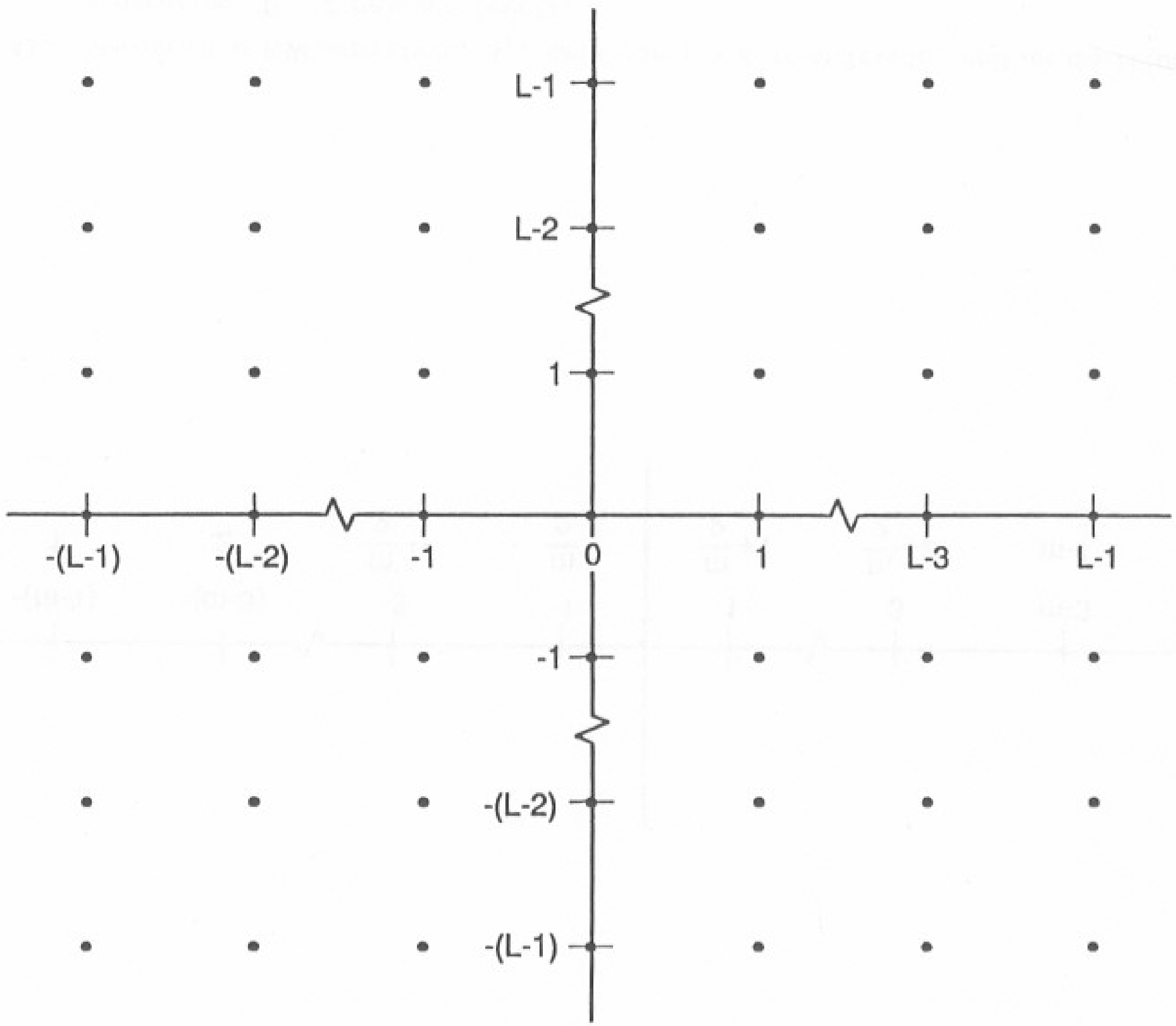

Figure A2. *Sketch of an M-QAM constellation, showing two m-PAM axes in quadrature, with a minimum amplitude separation, D=2, between signal states (indicated by dots). Note that by using figure A1, the number, M, of states in the constellation is $M=m^2$.*

Proakis (1989, Pg. 276) shows that

$$\langle A_i^2 \rangle = \frac{m^2-1}{3} = \frac{M-1}{3} \quad . \tag{A15}$$

Substitution of (A14) and (A15) into (A8) directly results in

$$\langle p_x^2 \rangle = 2\left[\frac{(3M-7)\,(M-1)}{15} + \frac{(M-1)^2}{9}\right]\left(\frac{\xi}{T}\right)^2 , \tag{A16}$$

which is also expression (9) in the main text.

REFERENCE

Proakis, J.G. (1989), *Digital Communications,* (McGraw-Hill Book Co. , Inc., New York, NY)

APPENDIX B

Derivation of the Mean-Square Transmitted Power for M-QPR

For M-QPR, consisting of a partial response (PR) modulation on each axis, the basic formulations for power, (A1) through (A8) of Appendix A, apply -- with the exception of the amplitude formulation (A2). The number of levels on each PR axis is 2L-1, as a result of the PR coding of an original, distinct L levels. If the L levels are equally likely, the 2L-1 levels will be triangularly distributed (Proakis, 1989, Pg. 544) with density function:

$$\frac{L-|i|}{L^2}, \quad i=-(L-1),\ -(L-2),\ \ldots,\ -1,0,1,\ \ldots,\ (L-2),\ (L-1)\ .$$

The amplitudes, A_i, are given by

$$A_i = -\ (L-i)\ , \tag{B1}$$

where the minimum spacing between signals in the signal constellation (see Figure B2) is taken as one "unit" (or D=1 as in equations (22a) and (22b) of the main text).[1] Thus,

$$\left\langle A_i^2 \right\rangle = \langle\ (L-i)^2\ \rangle\ , \tag{B2}$$

and

$$\left\langle A_i^4 \right\rangle = \langle (L-i)^4 \rangle\ .$$

---

[1] The arbitrary choice of minimum spacing makes no difference (e.g., D=1 or D=2) so long as the use is consistent throughout the evaluation of errors and their variability for a given modulation type.

However, because the zero amplitude occurs at i = L, and even numbered powers produce even symmetry about the origin (zero amplitude value), we can also state that, equivalently,

$$\left\langle A_i^2 \right\rangle = 2 \sum_{i=1}^{L-1} \frac{L-(L-i)}{L^2} (L-i)^2 + \frac{1}{L} \left\langle A_L^2 \right\rangle , \tag{B3}$$

or, noting that $A_L = 0$ in Figure B1,

$$\left\langle A_i^2 \right\rangle = \frac{2}{L^2} \sum_{i=1}^{L-1} i\,(L-i)^2 . \tag{B4}$$

Expression (B4) can be expanded into

$$\left\langle A_i^2 \right\rangle = 2 \sum_{i=1}^{L-1} i - \frac{4}{L} \sum_{i=1}^{L-1} i^2 + \frac{2}{L^2} \sum_{i=1}^{L-1} i^3 , \tag{B5}$$

and, after some algebra, reduced to

$$\left\langle A_i^2 \right\rangle = \frac{L^2-1}{6} . \tag{B6}$$

Also, we can state equivalently

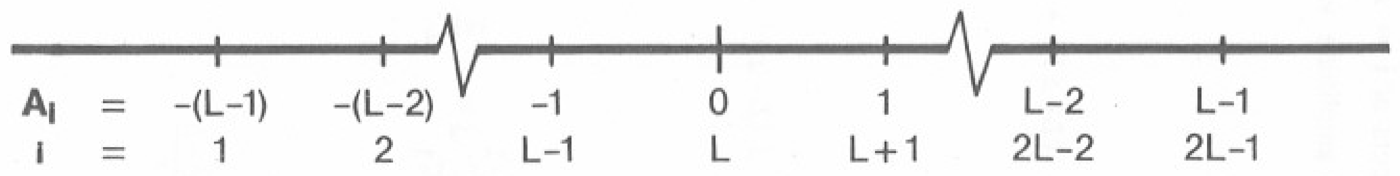

Figure B1. *Sketch of a partial response amplitude, A_i, axis for $i = 1$ to $2L - 1$ partial response coded levels of an originally uncoded L levels.*

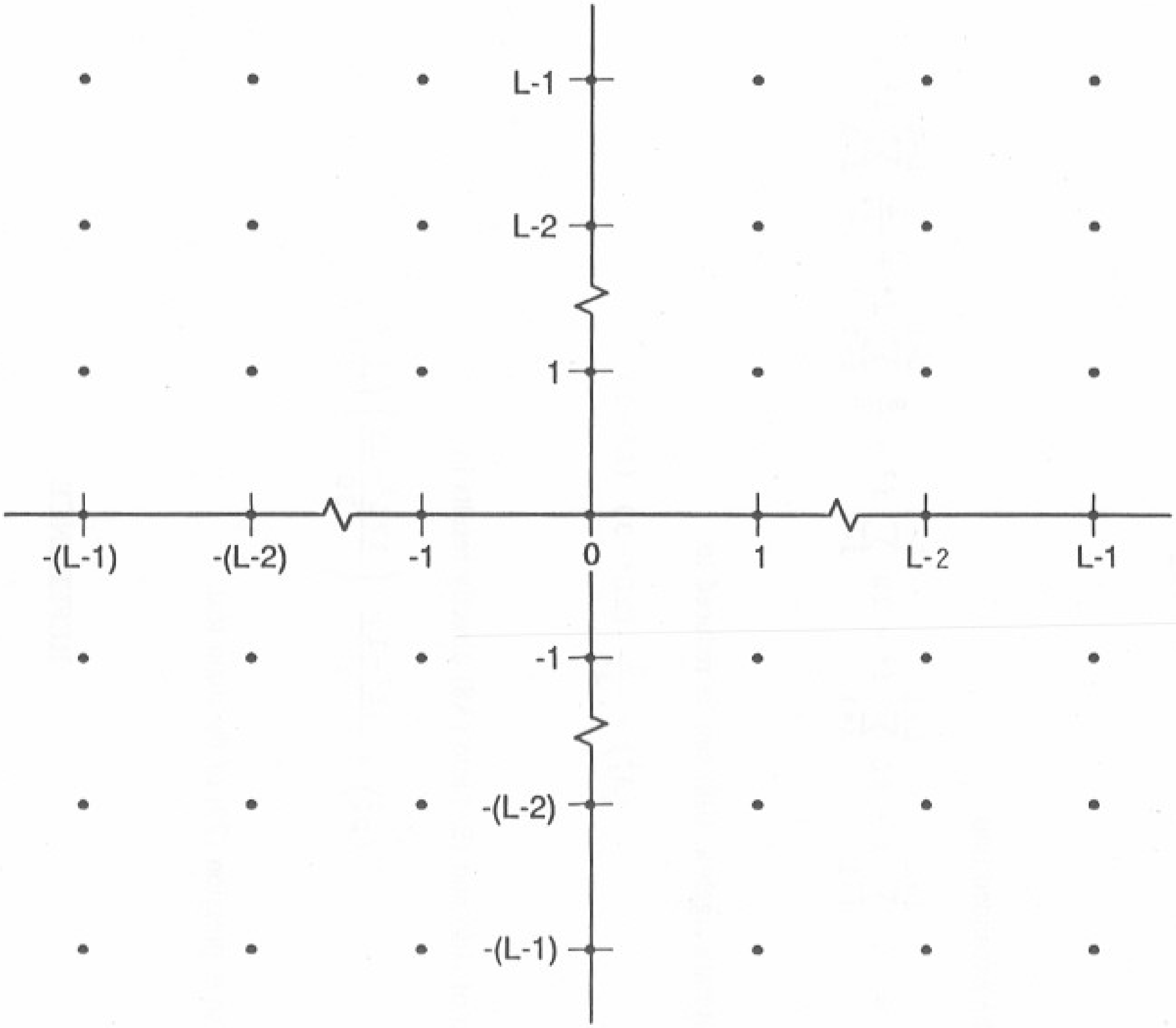

Figure B2. *Sketch of an M-QPR constellation, showing two partial response axes in quadrature, with a minimum amplitude separation, D=1, between signal states (indicated by dots). Note that by using figure B1, the number, M, of states in the constellation is $M = (2L-1)^2$.*

$$\left\langle A_i^4 \right\rangle = \frac{2}{L^2} \sum_{i=1}^{L-1} i \, (L-i)^4 \, , \tag{B7}$$

which can be expanded into

$$\left\langle A_i^4 \right\rangle = 2L^2 \sum_{i=1}^{L-1} i - 8L \sum_{i=1}^{L-1} i^2 + 12 \sum_{i=1}^{L-1} i^2 - \frac{8}{L} \sum_{i=1}^{L-1} i^4 + \frac{2}{L^2} \sum_{i=1}^{L-1} i^5 \, . \tag{B8}$$

After appropriate algebra, (B8) can be reduced to

$$\left\langle A_i^4 \right\rangle = \frac{1}{30} \, (2L^2-3) \, (L^2-1) \, . \tag{B9}$$

Substitution of (B6) and (B9) into (A8) directly results in

$$\left\langle p_x^2 \right\rangle = \frac{(L^2-1)}{3} \left(\frac{17L^2-23}{30} \right) \left(\frac{\xi}{T} \right)^2 , \tag{B10}$$

which is also expression (29) of the main text.

REFERENCE

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The error variability of received PSK (MSK), QAM and QPR modulations on digital microwave common carrier links resulting from transmitted power variations and the dispersive effects of terrain multipath using a two-ray model is analyzed. These results and an earlier prediction of long-term (annual) average symbol error rate (SYE) for the above modulations are used to predict median annual received symbol error rate. Since the average SYE was used earlier to represent median conditions, average and new median SYE's are compared.

Year-to-year variability of received power and BER of common carrier microwave transmissions caused by variation of annual refractivity gradient distributions on off-axis paths is also analyzed, and it is concluded that this variability is not significant vis-a-vis the annual variability discussed above.

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Digital modulation, error rate variability, microwave links, reception variability, refractivity gradient distributions, terrain multipath.

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